Hydrodynamic flow simulation in pipe networks including heat transfer

M. Töppel, M. Rada, E. Pasche

A finite element model for the solution of one-dimensional unsteady flow equations for pipe networks including heat transfer is presented. The differential equations for momentum, continuity and heat flow are discretized by one-dimensional finite elements. Since a nonlinear system of equations has to be solved iteratively, the Newton-Raphson-Method is applied.

An unsteady friction factor is implemented to improve the accuracy of modelling fast transients. Applying a time step control permits to simulate unsteady pipe flow with high resolution and very high accuracy in shorter calculation time. The length of each time step is readjusted according to the estimated calculation error.

1 Introduction

The majority of software products for the simulation of hydrodynamic flow in pipe networks use the method of characteristics (MOC) developed by Wylie and Streeter. Apart of several advantages, the MOC is an explicit method requiring very small time steps in order to meet the Courant criterion. For the simulation of pipe flow in a network with an average element length of about 1 m, e.g. to simulate flow in pipe networks of buildings, vehicles, ships or airplanes, a time step of less than 0.0008 s is required. Consequently 5 million time steps have to be calculated per simulated hour. Considering the computational expenses it is mandatory to use an implicit scheme for the solution of the differential equations to simulate flow scenarios of several hours.

An implicit finite element model for the solution of one-dimensional unsteady flow equations for pipe networks considering heat transfer is presented. An additional term is included in the momentum equation resulting from the consideration of an acceleration of the pipe network. This term might become important when pipe flow in accelerating objects like airplanes is modeled.

Computations with a quasi-steady friction term merely indicated satisfying results for slow changes of system variables, for which the wall shear stress exhibits a quasi-steady behavior. Experiments showed that rapid changes result in larger discrepancies between measured and simulated parameters due to the
steady or quasi-steady friction-coefficient. An unsteady friction factor is implemented into the numerical model to improve the accuracy of modeling fast transients.

2 Numerical model for the pipe network

2.1 Governing differential equations

The basis for the computation of one-dimensional flow in pressure pipelines is a differential equation system consisting of a momentum, a continuity and a heat flow equation. The flow equations can be written in the following form using the variables pressure $p$ and discharge $Q$:

$$\frac{\partial Q}{\partial t} + \frac{Q}{A} \frac{\partial Q}{\partial x} - \frac{Q}{\rho} \frac{\partial p}{\partial x} - \frac{Q^2}{A^2} \frac{\partial A}{\partial x} + gA \frac{\partial z}{\partial x} + \frac{A}{\rho} \frac{\partial p}{\partial x} + gAS_f - a_x A = 0$$

(1)

$$\frac{1}{\rho a^2} \left( A \frac{\partial p}{\partial t} + Q \frac{\partial p}{\partial x} \right) + \frac{\partial Q}{\partial x} - \frac{Q}{A} \frac{\partial A}{\partial x} = 0$$

(2)

where $g$ = acceleration of gravity; $z$ = coordinate of height; $S_F$ = friction slope; $\rho$ = density; $a$ = velocity of the pressure wave; $x$ = coordinate axis along the pipe; $t$ = time; $a_x$ = additional acceleration in x-direction.

The friction slope $S_F$ can be determined according to Darcy-Weisbach by the following expression using the friction coefficient $\lambda_F$:

$$S_F = \frac{\lambda_F}{d} \frac{Q|Q|}{A^2 2g}$$

(3)

Heat flow in pipes with negligence of the transversal dispersion yields to the following one dimensional differential equation of second order (Häfner et al.):

$$\lambda \frac{\partial^2 T}{\partial x^2} - \rho c_p \frac{Q}{A} \frac{\partial T}{\partial x} + k(T_w - T) = (\rho c_p)_s \frac{\partial T}{\partial t} - Q_v$$

(4)

where $T$ = fluid temperature; $\lambda$ = fluid heat conductivity; $c_p$ = specific heat capacity of the fluid; $k$ = heat transmission coefficient; $Q_v$ = independent source/sink term.

2.2 Consideration of friction loss

For the calculation of the friction coefficient $\lambda_F$ a formulation based on Zanke is implemented:

$$\lambda_F = (1 - \alpha) \frac{64}{\text{Re}} + \alpha \left( -2 \log \left( 2,7 \left( \frac{\log(\text{Re})}{\text{Re}} \right)^{1/2} + \frac{k_v/d}{3.71} \right) \right)^2$$

(5)
where $\alpha =$ weighing function; $Re =$ Reynolds number; $k_s =$ equivalent sand roughness; $d =$ pipe diameter. Advantages of equation (5) are the explicit calculation of the friction coefficient as well as to have a continuously differentiable function for both laminar and turbulent flow conditions.

Since many years researchers have developed approaches for an unsteady friction factor. ZIELKE analytically derived a term for laminar flow conditions. The friction term is related to the instantaneous average flow velocity and to weighted past velocity changes.

BRUNONE developed a model relating the unsteady part of the friction coefficient to the instantaneous local acceleration $\partial v / \partial t$ and to the instantaneous convective acceleration $\partial v / \partial x$. The original model however exhibits weak points, since it fails to predict the correct sign of the convective term for particular flow and wave directions in acceleration and deceleration phases. The corrected formula (PEZZINGA), transformed into a flow equation by substituting $v$ by $Q/A$, yields:

$$\lambda_{\text{unsteady}} = \lambda_{\text{steady}} + \frac{2kd}{Q|Q|}\left(\frac{A}{\partial t} - Q\frac{\partial A}{\partial t} + a \cdot \text{sgn}(Q)\left\|A\frac{\partial Q}{\partial x} - Q\frac{\partial A}{\partial x}\right\|\right)$$

where $k =$ BRUNONE friction coefficient.

### 2.3 Source/sink term of the heat flow equation

The heat flux $\dot{Q}$ [W] through the pipe wall can be described with FOURIER’S law of heat conduction $\dot{Q} = k'(T_a - T_i)$ where $T_a =$ ambient temperature; $T_i =$ fluid temperature; $k' =$ thermal conductivity.

If the heat flux is divided by the fluid volume $V = A \cdot 1$, $\dot{Q}_v$ [W/m³] is the heat flux density. Substituting $k'/V$ by the thermal conductivity referring to the fluid volume $k$, FOURIER’S law for a circular pipe yields:

$$\dot{Q}_v = \frac{4/d_i^2}{\alpha d_i + \frac{1}{2\lambda_i} \ln \frac{d_i}{d_o} + \frac{1}{\alpha_s d_s}} (T_a - T_i)$$

where $d =$ diameter; $\alpha =$ heat transfer coefficient; $\lambda =$ heat conductivity of pipe material. The heat transfer coefficient at the outer pipe wall can be described for inactive air according to VDI-Wärmeatlas by $\alpha_a = 8 + 0.04 \cdot \Delta T$ [W/m²/K], the coefficient at the inner pipe wall mainly depends on the NUSSELT number which is a function of the REYNOLDS number and the PRANDTL number, the fluid heat conductivity and a characteristic length. It can be expressed by the relation $\alpha_i = Nu \cdot \lambda / L$. There is a variety of formulas describing the NUSSELT number for pipe flow which are valid for different REYNOLDS and PRANDTL numbers.
2.4 Solving the differential equations with the method of finite elements

The method of finite elements is used for the simulation of unsteady pipe flow considering heat transfer. The variables to be determined are pressure $p_{K,i}$, discharge $Q_{K,i}$ and fluid temperature $T_{K,i}$ at every node. The differential equations can be applied to single pipelines. At junctions a discontinuity of discharge is encountered that cannot be calculated by one-dimensional finite elements. A solution is found by the use of algebraic equations for junctions as well as for all special elements of the pipe-network, e.g. for pumps, valves, tanks etc.

Another characteristic of the differential equations is the non-linearity. For this reason the solution of the finite element formulation of the resulting set of equations is received with the iterative Newton-Raphson-procedure which can be represented in the following form:

$$
\begin{bmatrix}
  x_{1,n+1} \\
  x_{2,n+1} \\
  \vdots \\
  x_{m-1,n+1} \\
  x_{m,n+1}
\end{bmatrix} = \begin{bmatrix}
  x_{1,n} \\
  x_{2,n} \\
  \vdots \\
  x_{m-1,n} \\
  x_{m,n}
\end{bmatrix} - \begin{bmatrix}
  \frac{\partial F_1}{\partial x_{1,n}} & \cdots & \frac{\partial F_1}{\partial x_{m,n}} \\
  \vdots & \ddots & \vdots \\
  \frac{\partial F_m}{\partial x_{1,n}} & \cdots & \frac{\partial F_m}{\partial x_{m,n}}
\end{bmatrix}^{-1}
\begin{bmatrix}
  F_{1,n} \\
  F_{2,n} \\
  \vdots \\
  F_{m-1,n} \\
  F_{m,n}
\end{bmatrix}
$$

where $F_{i,n}$ = finite element formulation of the differential and algebraic equations for each node; $x_{i,n+1}$ = solution of the variables for the next iteration step. Therefore it is necessary to calculate the derivatives of the finite element formulation for the differential equations.

The model uses a formulation based on the method of weighted residuals. In this method the differential equations are applied in integral notation. The shape functions $N_n$ of the finite element approximation are used as weighing parameters $N_W$. The contribution of a single element of length $L$ is defined as:

$$
f_1 = \int_0^L N_W \left( \frac{\partial Q}{\partial t} + \frac{Q \partial Q}{A \partial x} - \frac{Q \partial A}{A \partial P} \frac{\partial P}{\partial t} - \frac{Q^2}{A^2} \frac{\partial A}{\partial x} + \frac{A \partial P}{\rho \partial x} + gA \frac{\partial P}{\partial x} + gAS_p - a_x A \right) dx \quad (9)
$$

$$
f_2 = \int_0^L N_W \left( \frac{\partial Q}{\partial x} - \frac{Q \partial A}{A \partial x} + \frac{A}{\rho a_x} \left( \frac{\partial P}{\partial x} + \frac{Q}{A} \frac{\partial P}{\partial x} \right) \right) dx \quad (10)
$$

$$
f_3 = \int_0^L N_W \left( \lambda \frac{\partial^2 T}{\partial x^2} - \rho c_p Q \frac{\partial T}{A \partial x} + k (T_w - T) - (\rho c_p)_s \frac{\partial T}{\partial t} + \dot{Q}_v \right) dx \quad (11)
$$

A major advantage of the integral formulation of the method of finite elements is the possibility of reducing the order of the derivatives inside the integral from two to one by integrating by parts. Therefore equation (11) yields:
To solve the equation system with the Newton-Raphson-scheme equation (9), (10) and (12) are differentiated to pressure, discharge and temperature at all nodes n. This leads to an enormous effort and to large matrices. Since the flow equations (9) and (10) are only slightly dependent on equation (12), it is possible to calculate two separate equation systems during the iteration process and update values after each iteration step. Therefore the calculation costs are significantly reduced and the flow equations only need to be differentiated to discharge and flow while equation (12) is differentiated to temperature.

With the relations \( \frac{\partial y}{\partial y_n} = N_n, \frac{\partial f(y)}{\partial y_n} = N_n \frac{\partial f(y)}{\partial y} \) and \( \frac{\partial^2 y}{\partial y_n \partial x} = \frac{\partial N_n}{\partial x} \), the derivatives are:

\[
\frac{\partial f_1}{\partial p_n} = \int_0^1 N^W N_n \left\{ \frac{Q}{A^2} \frac{\partial A}{\partial p} \left( \frac{\partial A}{\partial t} - \frac{\partial Q}{\partial t} \right) - \frac{Q}{A} \frac{\partial N_n}{\partial x} + \frac{Q}{A} \frac{\partial A}{\partial t} \right\} dx \tag{13}
\]

\[
\frac{\partial f_2}{\partial Q_n} = \int_0^1 N^W N_n \left\{ \frac{A}{A^2} \frac{\partial A}{\partial p} \left( \frac{\partial A}{\partial t} - \frac{\partial Q}{\partial t} \right) + \frac{Q}{A} \frac{\partial N_n}{\partial x} - \frac{Q}{A} \frac{\partial A}{\partial t} \right\} dx \tag{14}
\]

\[
\frac{\partial f_3}{\partial p_n} = \int_0^1 N^W N_n \left\{ \frac{1}{N_n} \frac{\partial N_n}{\partial x} + \frac{1}{\rho a^2} \frac{\partial p}{\partial x} - \frac{1}{A} \frac{\partial A}{\partial x} \right\} dx \tag{15}
\]

\[
\frac{\partial f_4}{\partial Q_n} = \int_0^1 N^W N_n \left\{ \frac{1}{N_n} \frac{\partial N_n}{\partial x} + \frac{1}{\rho a^2} \frac{\partial p}{\partial x} - \frac{1}{A} \frac{\partial A}{\partial x} \right\} dx \tag{16}
\]

\[
\frac{\partial f_5}{\partial T_n} = \int_0^1 N^W N_n \left\{ \frac{1}{N_n} \frac{\partial N_n}{\partial x} + \frac{1}{\rho a^2} \frac{\partial p}{\partial x} - \frac{1}{A} \frac{\partial A}{\partial x} \right\} dx + BC \tag{17}
\]

For the time integration scheme a finite difference expression is used:

\[
\frac{\partial z(t+\Delta t)}{\partial t} = \theta \left( \frac{z(t+\Delta t) - z(t)}{\Delta t} \right) + (1-\theta) \frac{\partial z(t)}{\partial t} \tag{18}
\]

where \( \theta \) = relaxation coefficient; \( z \) = variable. From this expression equation (19) results that is used to calculate the derivatives of the finite element formulation:
\[
\frac{\partial}{\partial z_n (t+\Delta t)} \left( \frac{\partial z(t+\Delta t)}{\partial t} \right) = N_n(x) \frac{\theta}{\Delta t}
\]  

(19)

The extreme complexity of equations (9) to (17) makes an exact integration impossible. Therefore, the integrals are solved by numerical integration, e.g. by GAUSS-LEGENDRE integration.

3 Numerical and experimental results

3.1 Pressure surges

The differential equation system is used to calculate pressure surges in pipe-networks. Pressure surges arise for example after fast closing or opening of valves as well as switching on or off pumps. The reason is the mass inertia of the fluid.

![Figure 1](image-url) Architecture of the experimental test rig (not to scale)

To validate the numerical model measurements are compared to numerical results. At the department of river and coastal engineering of Hamburg University of Technology a physical model of an aircraft potable water system was assembled using original aircraft components (Fig. 1). It consists of two tanks, each 375 l, a compressor unit, 3/4” and 1/2” stainless steel pipes and valves representing water faucets, toilet rinse valves and coffee maker. The maximum length from the tanks to the most distant valve is approximately 70 m.

Pressure and discharge at different points of the water system are recorded with a frequency of 500 Hz using the pressure sensor “S-11 REL” from Greisinger and the flow meter “Promag 53” from Endress+Hauser.

Several water hammer experiments were run to compare numerical and measured results. Starting with a pressure of 2 bar, the valve is opened after 10 s and closed again after 5 s. When the valve is opened the pressure changes until
steady flow conditions have developed. By closing the valve a transient event is initiated and the water hammer wave travels between the closed valve and the tank. After the first pressure wave has passed, the pressure decreases to the vapor pressure of the liquid. At this point cavitation occurs. Computational results, considering an unsteady friction factor, agree well to the measured values (Fig. 2). Since the pressure sensor only records positive relative pressures, the pressure drop down to vapor pressure during water hammer is not registered.

![Figure 2](image)

**Figure 2** Comparison of measured and calculated pressure

### 3.2 Transient heat conduction

Assuming a pipe with no heat conduction through the wall, an analytical solution of the heat flow equation can be derived. The assumptions made are only realistic for geometries where the cross section exceeds the length. Still this example can demonstrate the accuracy of the numerical solution. For the analytical solution of the heat flow equation

\[
\lambda \frac{\partial^2 T}{\partial x^2} = \rho c \frac{\partial T}{\partial t} \tag{20}
\]

**WEIGAND** derived the expression

\[
T = T_0 + (T_i - T_o) \left( \frac{x}{L} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin \left( \frac{n\pi}{L} x \right) \exp \left( -\frac{(n\pi)^2 \frac{\lambda}{L^2 \rho c} t} \right) \right) \tag{21}
\]

The differential equations are solved by applying the characteristic parameters of water, assuming no flow and a thermal conductivity \( k = 0 \). The boundary condition for steady state is a fluid temperature of 10°C that decreases to 0°C immediately after starting the transient calculation.
The calculated temperature distribution for different times is almost identical to the data analytically derived (Fig. 3). This exemplarily demonstrates that the numerical scheme is appropriate to solve the differential equations.

![Figure 3](image)

**Figure 3** Comparison of analytically derived and numerical calculated temperatures

## 4 Conclusion

With the presented numerical methods it is possible to simulate unsteady pipe flow with a high resolution and a very satisfying accuracy in short calculation time.

The implicit finite element scheme allows larger time steps than the MOC. To minimize the costs of calculation, it is necessary to use an adaptive time step control. The length of each time step is readjusted according to the estimated calculation error. Using the method of finite elements the element length $\Delta x$ can be adjusted to the pipe geometry and it is possible to use a different number of elements for the iteration of the flow equation than for the heat flow equation.

The numerical solution with the finite element method should satisfy convergence and stability criteria. The convergence of the presented model was examined for a number of different $\Delta t$ and $\Delta x$ sizes. The method showed high accuracy and convergence as well as a numerically stable behavior.

To simulate flow scenarios of several hours particularly close-to-reality the presented algorithm should be applied for solving the one-dimensional flow and heat flow equations. The consideration of an unsteady friction factor leads to more accurate solutions regarding cases of water hammer. In addition, the adaptive time step control reduces calculation costs dramatically. Therefore, this algorithm is particularly suited for optimization as well as case studies requiring
many simulations with changing parameters. So far the presented numerical model was successfully used for simulating pipe flow in airplanes. Of course it is also applicable to other fields of interest, e.g. the calculation of pressure surges in tall buildings, in industrial plants or in urban water supply networks as well as the calculation of the exact size of a surge tank or air chamber with varying boundary conditions.

5 References


VDI [Hrsg.]: VDI-Wärmeatlas, Berechnungsblätter für den Wärmeübergang, 9. Auflage, Springer Verlag, Berlin [u.a.], 2002


Authors:

Dipl.-Ing. Markus Töppel,
Dipl.-Ing. Martin Rada,
Prof. Dr.-Ing. Erik Pasche
Technische Universität Hamburg-Harburg
Institut für Wasserbau
Denickestraße 22
21073 Hamburg
Tel.: ++49 – 40 – 42878-4274
Fax: ++49 – 40 – 42878-2802
m.toeppel@tu-harburg.de
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