Numerical analysis of embankment erosion caused by overflow using shallow water equations

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Background

Overflow and piping are the primary causes of failures of embankments such as dams, levees and irrigation ponds.

1. **Overflow**
   - 70 to 80% of levee failures have been caused by overflow.
   - About 50% of dam breaks have been caused by overflow.

2. **Piping**
   - 40% of causes of dam breaks

3. **Slide**
   - a few percents

![Diagram showing the causes of failures in embankments]
Cases of irrigation ponds

Irrigation ponds which failed due to overflow in 2004, Awaji Island, Japan

Plunge pool

Loss of vegetation
Case of a levee

A levee suffering overflow in 2004, Fukui pref., Japan
Previous studies

Visser (1998) and Coleman et al. (2002) investigated the breaching process of cohesionless embankments during overtopping failure.

Zhu (2006) focused his investigation on the failure process of cohesive embankments.

Hanson et al. (2005) conducted large-scale overflow-embankment tests using silty sand and a clayey material, and Hanson et al. (2011) integrated the material properties for embankment breach. They have intensively been working on the headcut advance, too.

Numerical and theoretical studies on this topic have been reported as well. (e.g., Zerrouk & Marche 2004; Wang & Bowles 2006; Wang & Bowles 2007; Faeh 2007; Fujisawa et al. 2009)
When does the headcut appear?

Cohesionless material


Cohesive material

Subcritical region: changes in flow can propagate both upstream and downstream.

Supercritical region: changes in flow propagate only downstream.

Where the friction stress reaches the critical stress and the erosion start.
Erosion on supercritical region

The upstream embankment does not change.
Erosion on subcritical region

The entire profile of the embankment is changing.
Governing equations

2D shallow water equations

\[ \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S \]

Finite Volume Method

Temporal change of erosion surface

\[ \frac{\partial z}{\partial t} = - \frac{E(U)}{1 - \lambda_p} \]

\[ U = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix}, \quad F = \begin{pmatrix} uh \\ u^2 h + gh^2 / 2 \\ uvh \end{pmatrix}, \quad G = \begin{pmatrix} vh \\ uvh \\ v^2 h + gh^2 / 2 \end{pmatrix} \]

\[ S = \begin{pmatrix} 0 \\ - gh(\partial z / \partial x + S_{fx}) \\ - gh(\partial z / \partial y + S_{fy}) \end{pmatrix} \]

Erosional surface

Water flow

Embankment
Erosion of soils

Water pressure

Bed shear stress (Tractional stress)

Erosion rate

\[ E = \alpha (\tau - \tau_c)^\gamma \]

Erosion rate: \( E \)  
Applied shear stress: \( \tau \)  
Material constants: \( \alpha, \gamma \)  
Critical shear stress: \( \tau_c \)  

Volume of soils eroded from unit surface area during unit time
Finite Volume Approach

Area integral

\[ \int_{A_i} \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} \, dA = \int_{A_i} S \, dA \]

Divergence theorem of Gauss

\[ \frac{d}{dt} \int_{A_i} U \, dA = - \int_{\Gamma} E^* \, d\Gamma + \int_{A_i} S \, dA \]

Intercell normal flux \( E^* = Ft_x + Gt_y \)

Finite volume approximation for each cell

\[ \frac{dU}{dt} = \frac{1}{A_i} \sum_j E^* \Delta \Gamma_{ij} + S \]

\( A_i \): Area of \( i \)th cell

\( \Gamma_{ij} \): Length of \( j \)th edge of \( i \)th cell

The ordinary differential equation at each cell is solved with a time-integration scheme.
The conservative variables of the flow depth ($h$) and the flow rate ($hu$, $hv$) are stored at the cell centers.

The height of the flow bed ($z$) is stored and computed at the cell vertices (= cell nodes).
1. The numerical method applied to this problem is MUSCL-type finite volume method, using an approximate Riemann solver with the implementation of a slope limiter and Surface Gradient Method (SGM).

   “Riemann solver” means the computation of exact solutions for the intercell flux.

   “Slope limiter” constrains the gradients of variables on cells for stable computation.

2. The change in flow bed height is simultaneously solved with the shallow water equations. TVD Runge-kutta scheme is used for the time integration.

3. Fictitious cells were installed at the boundary to input natural boundary condition for water flow and to accurately compute the elevation change of the water bed.
Treatment of source term & Time integration

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S
\]

\[
\frac{\partial U_i}{\partial t} = -\frac{1}{A_i} \sum_{j=1}^{3} E \cdot n_{ij} \Delta \Gamma_{ij} + S_i
\]

\[
\begin{cases}
\frac{\partial U_i}{\partial t} = S_f \quad \text{Implicitly solved} \\
\frac{\partial U_i}{\partial t} = -\frac{1}{A_i} \sum_{j=1}^{3} E \cdot n_{ij} \Delta \Gamma_{ij} + S_0 \quad \text{Explicitly solved}
\end{cases}
\]

Third order TVD Runge-Kutta Scheme

\[
\frac{\partial z}{\partial t} = -\frac{E}{1 - \lambda}
\]

\[
U^{(1)} = U^n + \Delta t L(z^n, U^n)
\]

\[
U^{(2)} = \frac{3}{4} U^n + \frac{1}{4} U^{(1)} + \frac{1}{4} \Delta t L(z^{(1)}, U^{(1)})
\]

\[
U^{n+1} = \frac{1}{3} U^n + \frac{2}{3} U^{(2)} + \frac{2}{3} \Delta t L(z^{(2)}, U^{(2)})
\]

\[
S = S_0 + S_f = \begin{pmatrix} 0 \\ g h S_{ox} \\ g h S_{oy} \end{pmatrix} + \begin{pmatrix} 0 \\ -g h S_{fx} \\ -g h S_{fy} \end{pmatrix}
\]

\[
U = \begin{pmatrix} h \\ u h \\ v h \end{pmatrix}, \quad F = \begin{pmatrix} u^2 h + g h^2 / 2 \\ uvh \\ v^2 h + g h^2 / 2 \end{pmatrix}, \quad G = \begin{pmatrix} v h \\ uvh \\ v^2 h + g h^2 / 2 \end{pmatrix}
\]
The conservative variables are the flow depth (h) and the flow rates (hu, hv) and usually these variables are interpolated over cells. However, the SGM interpolates the height of water surface instead of the flow depth.

Surface Gradient Method (C-property)

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S
\]

\[
U = \begin{pmatrix} h \\ uh \\ vh \end{pmatrix}, \quad F = \begin{pmatrix} uh \\ u^2h + gh^2/2 \\ uvh \end{pmatrix}, \quad G = \begin{pmatrix} vh \\ uvh \\ v^2h + gh^2/2 \end{pmatrix}
\]

\[
S = S_0 + S_f = \begin{pmatrix} 0 \\ ghS_{ox} \\ ghS_{oy} \end{pmatrix} + \begin{pmatrix} 0 \\ -ghS_{fx} \\ -ghS_{fy} \end{pmatrix}
\]

When the flow velocities are zero and the water level is constant;

\[
\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = S_0
\]

\[
F = \begin{pmatrix} 0 \\ gh^2/2 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ gh^2/2 \end{pmatrix}, \quad S_0 = \begin{pmatrix} 0 \\ -gh\partial z/\partial x \\ -gh\partial z/\partial y \end{pmatrix}
\]

Flux term

\[
g \frac{1}{2} \frac{\partial}{\partial x} \left( \eta - z \right)^2
\]

\[
= g \frac{1}{2} \frac{\partial}{\partial x} \left( \eta^2 - 2\eta z + z^2 \right)
\]

\[
= g \left[ -\eta \frac{\partial z}{\partial x} + \frac{1}{2} \frac{\partial z^2}{\partial x} \right]
\]

Gravitational term

\[
gh \frac{\partial z}{\partial x} = g(\eta - z) \frac{\partial z}{\partial x}
\]

\[
= g\eta \frac{\partial z}{\partial x} - z \frac{\partial z}{\partial x}
\]

\[
= g\eta \frac{\partial z}{\partial x} - \frac{1}{2} \frac{\partial z^2}{\partial x}
\]
2-D simulation - examples -

Initial water surface

Inflow
$1.5 \text{[m}^2/\text{s}]$
Verification

Input parameters:

Porosity=0.395, Critical shear stress ($\tau_c$)=1.0Pa, $\alpha$=7.84E-5 m/s/Pa$^{3/2}$

$\gamma$=1.5, Manning’s coefficient=0.0168

Boundary conditions:

Inflow rate=0.021 m$^3$/s/m at the left side.

Free outfall boundary for the right side.
A scar (initial disturbance) with the depth of 2 cm is given on the top of the embankment, which might induce the concentration of the overflow and erosion.
Concept on 3-D simulation

Input parameters:
Porosity=0.395, Critical shear stress ($\tau_c$)=0.1Pa, $\alpha=8.42\text{E-5 m/s/Pa}^{3/2}$
$\gamma=1.5$, Manning’s coefficient=0.0158

Boundary conditions:
Inflow rate=0.0029 m$^3$/s/m at the left side.
Free outfall boundary for the right side.
Numerical results

20 sec

100 sec

50 sec

600 sec
Summary

1. The numerical solution of 2D shallow water equations and the erosion rates of embankment material allow us to determine the profiles of embankments subjected to overflow erosion.

2. Surface Gradient Method is quite helpful when the complex geometry of flow beds is dealt with. (We must check “C-property” that makes sure the balance between the flux and gravitational terms under steady state)

3. 3-dimensional analysis is also possible by the proposed method. The next step is to verify and predict the width of the erosion channel and to try the large scale simulation.
Thank you for your attention!
Slope limiter
Fictitious cell

Left side

Right side

left limited gradient

centroid