Sixth International Conference on Scour and Erosion
Paris, August 27-31, 2012

Hydro-mechanical analysis of internal erosion with mass exchange between solid and water

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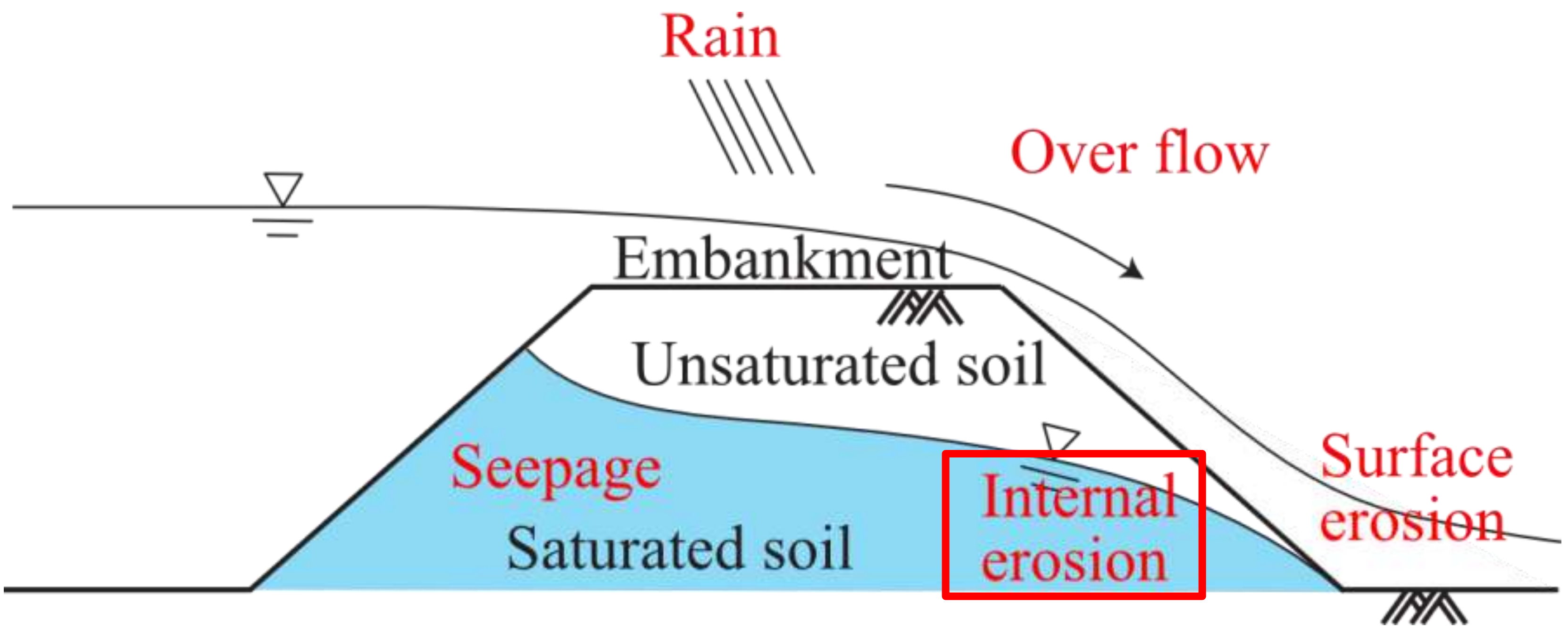
Failure due to seepage and overtopping

Sanjo, Niigata (MLIT, 2004)

Seepage and overtopping

- Internal erosion due to seepage leading to piping
- Surface erosion due to overtopping leading to breaching
- Combined effects of seepage and overtopping on failure



Hydro-mechanical coupled analysis of IVB problem based on unsaturated soil mechanics

Hydro-mechanical analysis

- Porous media theory at finite strain (de Boer 2000, Schrefler 2002)
  - Soil skeleton, pore water and pore air
  - Static, quasi-static and dynamic problem
- Balance equations
  - Mass and momentum balances of soil skeleton
    - Internal erosion = mass exchange between soil skeleton and pore water
  - Mass and momentum balances of pore water and air
- Constitutive models
  - Bishop's effective stress (average skeleton stress)
  - Darcy flow: interaction between soil skeleton and fluid
  - Compressibility of pore water
  - Compressibility of pore air
  - Water retention curve (WRC)
    - van Genuchten (VG) model
  - Constitutive model for soil skeleton
    - Linear elastic model
  - Internal erosion
    - Change in fine content dependent on Darcy flow velocity

Balance laws in Porous Media Theory

- Mass balance (local form)

$$\frac{D^\alpha \rho^\alpha}{Dt} + \rho^\alpha \,\mathrm{div}\,\mathbf{v}^\alpha = \hat{\rho}^\alpha$$

$$\sum_{\alpha=1}^{3} \hat{\rho}^\alpha = 0$$

ρ^α : partial density

$\mathbf{v}^\alpha$: velocity

$\hat{\rho}^\alpha$: mass exchange (density rate)

Internal erosion = mass exchange between soil skeleton and pore water

- Linear momentum balance (local form)

$$\mathrm{div}\,\boldsymbol{\sigma}^\alpha + \rho^\alpha \mathbf{b} + \hat{\mathbf{p}}^\alpha = \mathbf{0} \qquad \sum_{\alpha=1}^{3} \hat{\mathbf{p}}^\alpha = \mathbf{0}$$

$\boldsymbol{\sigma}^\alpha$: Cauchy partial (averaged) total stress tensor of α

$\mathbf{b}$: body force vector of α

$\hat{\mathbf{p}}^\alpha$: interaction force vector of α

Constitutive equations for internal erosion

- Mass exchange (density rate) related to fine content rate

$$\hat{\rho}^{\mathrm{s}} = (1-n)\rho^{\mathrm{sR}} \frac{D^{\mathrm{s}} f_{\mathrm{c}}}{Dt}$$

$$\hat{\rho}^{\mathrm{w}} = -(1-n)\rho^{\mathrm{sR}} \frac{D^{\mathrm{s}} f_{\mathrm{c}}}{Dt}$$

$$\hat{\rho}^{a} = 0$$

n : porosity

ρ^{sR} : intrinsic density of particle

f_{c} : fine content

volumetric ratio of fine particle to soil particle

- Fine content rate dependent on Darcy flow velocity and ultimate fine content (Sterpi 2003, Cividini et al. 2009)

$$\frac{D^{\mathrm{s}} f_{\mathrm{c}}}{Dt} = -\mu_{\mathrm{e}}(f_{\mathrm{c}} - f_{\mathrm{c}\infty})\left|\frac{ns^{\mathrm{w}}\mathbf{v}^{\mathrm{ws}}}{v_{\mathrm{ref}}}\right|^{\psi_{\mathrm{e}}} \quad (f_{\mathrm{c}} > f_{\mathrm{c}\infty})$$

$$\frac{D^{\mathrm{s}} f_{\mathrm{c}}}{Dt} = 0 \quad (f_{\mathrm{c}} \le f_{\mathrm{c}\infty})$$

$ns^{\mathrm{w}}\mathbf{v}^{\mathrm{ws}}$: Darcy flow velocity

$f_{\mathrm{c}\infty}$: Ultimate fine content

Validation of erosion model

- Internal erosion test (Sterpi 2003)

Sand and silt
Fc =23%

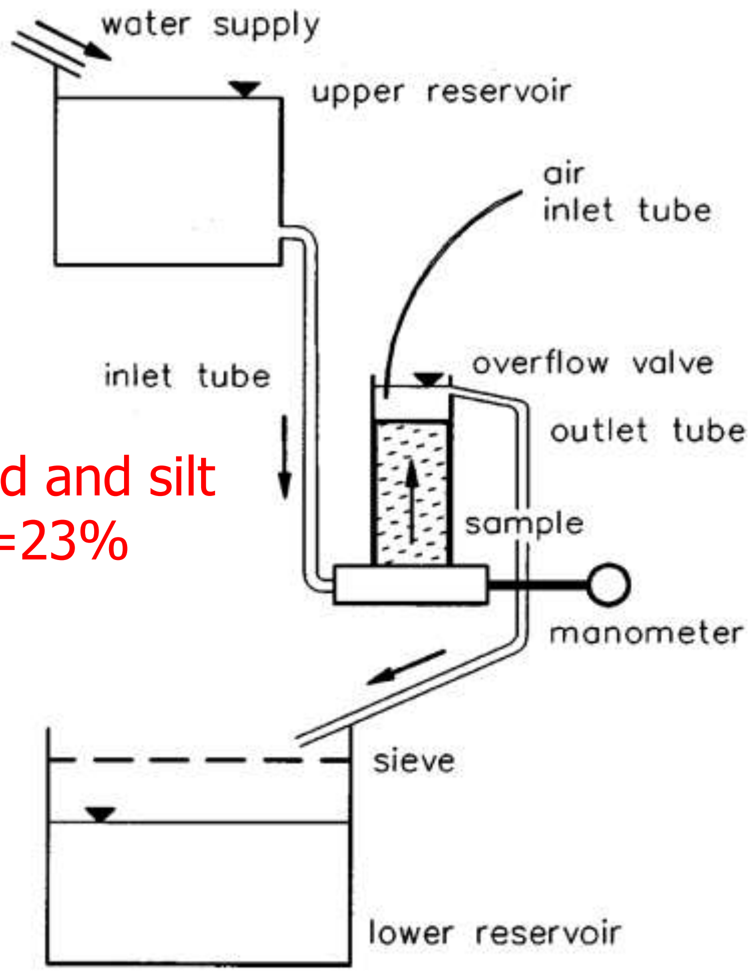
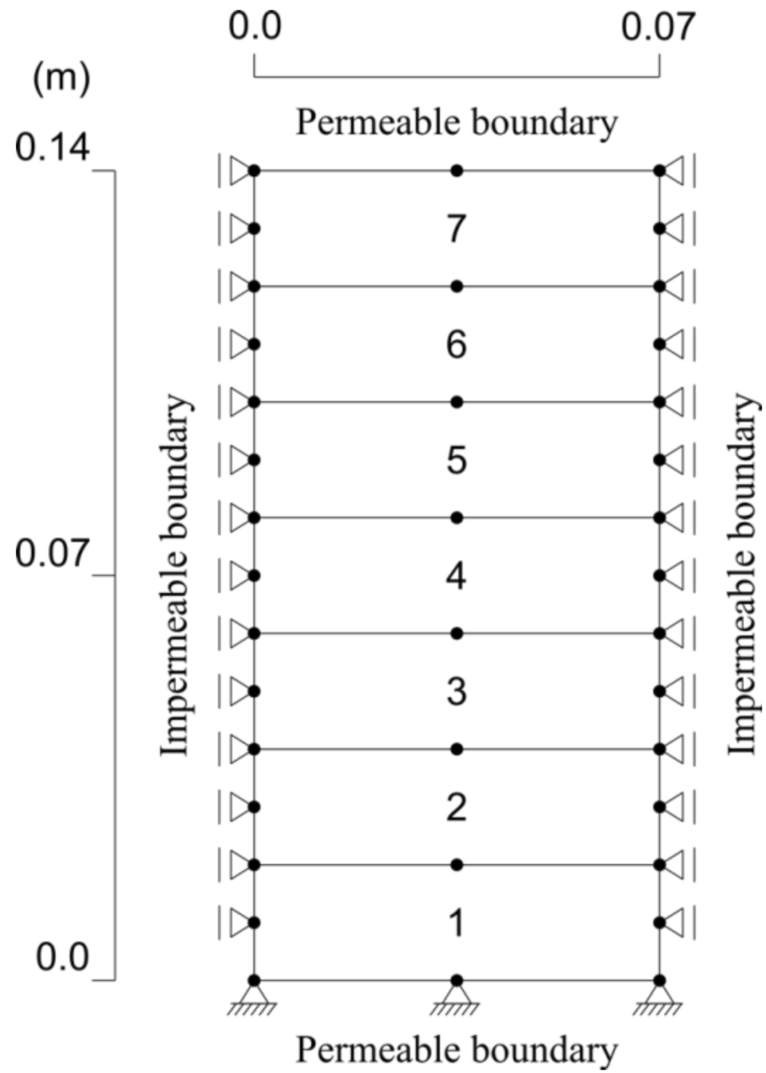
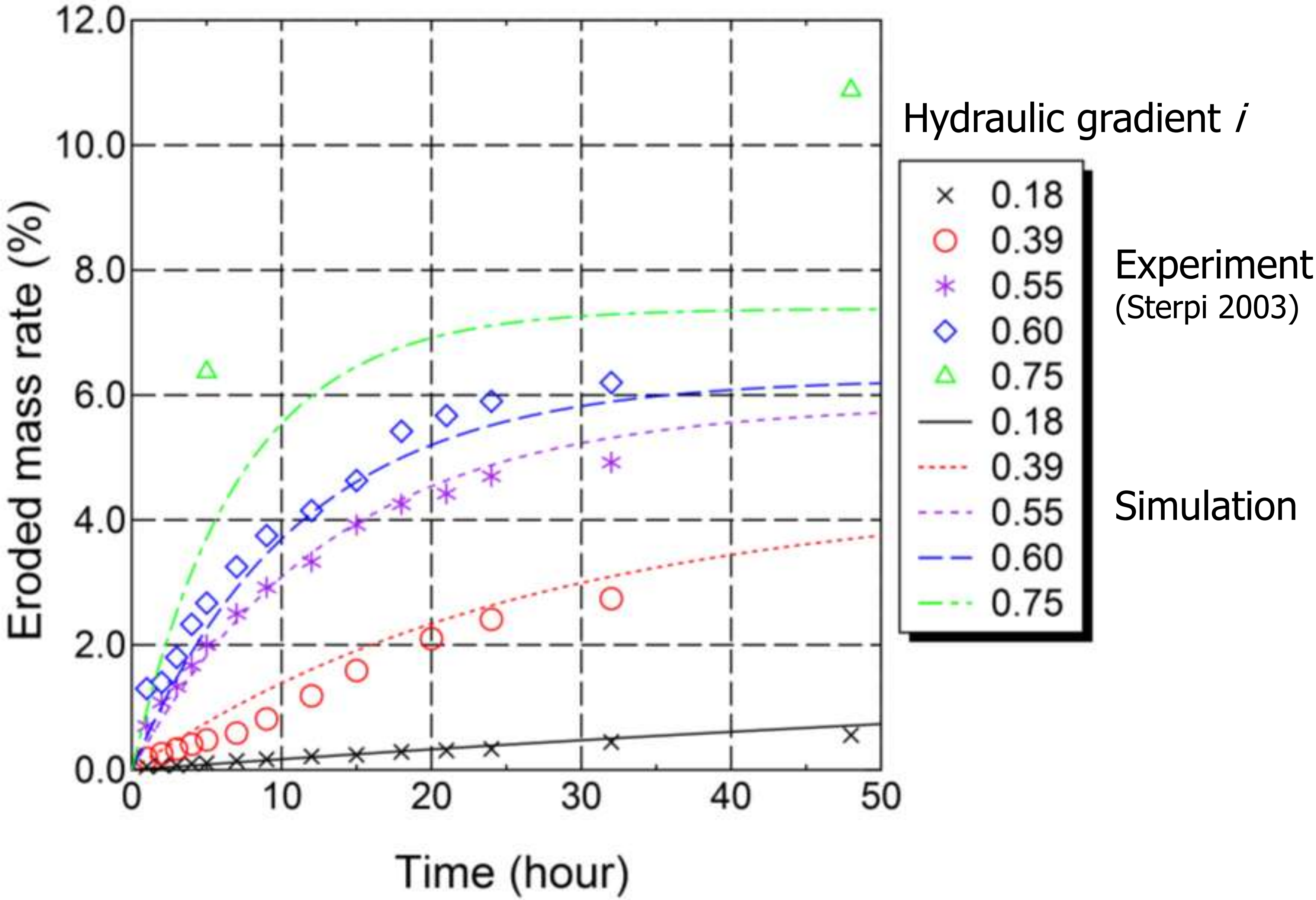


Fig. 3. Experimental setup for seepage tests



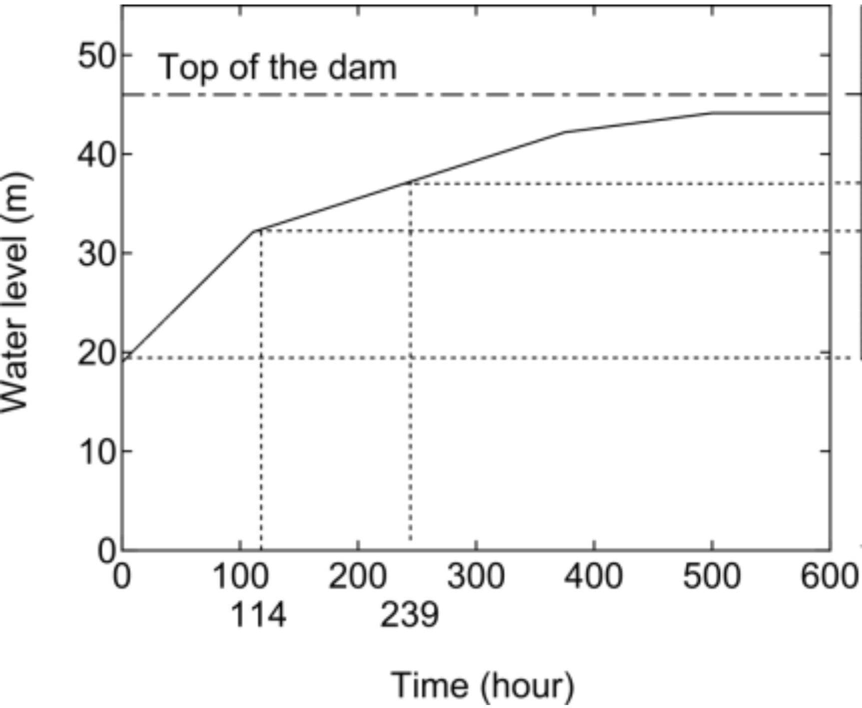
Eroded mass rate with different *i*

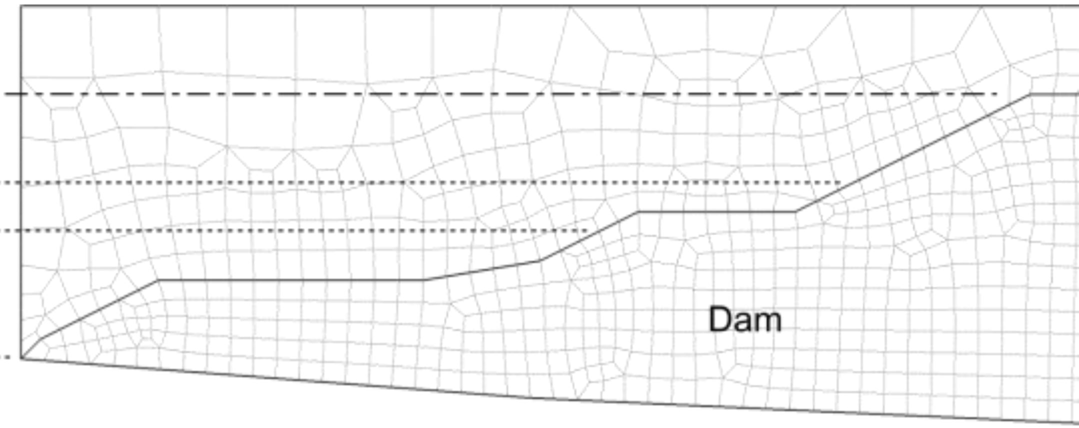


Formation of water flow path in landslide dam

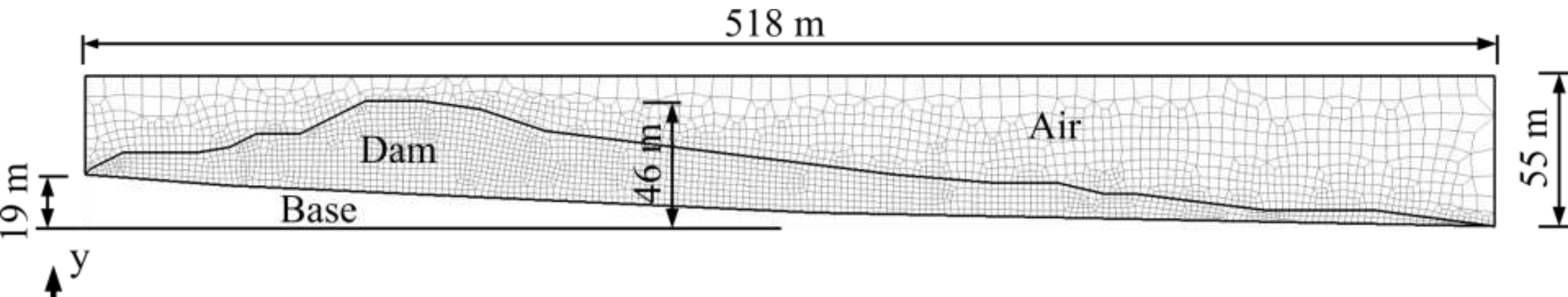
Yunokura landslide dam after the 2008 Iwate-Miyagi Nairiku earthquake

Numerical conditions



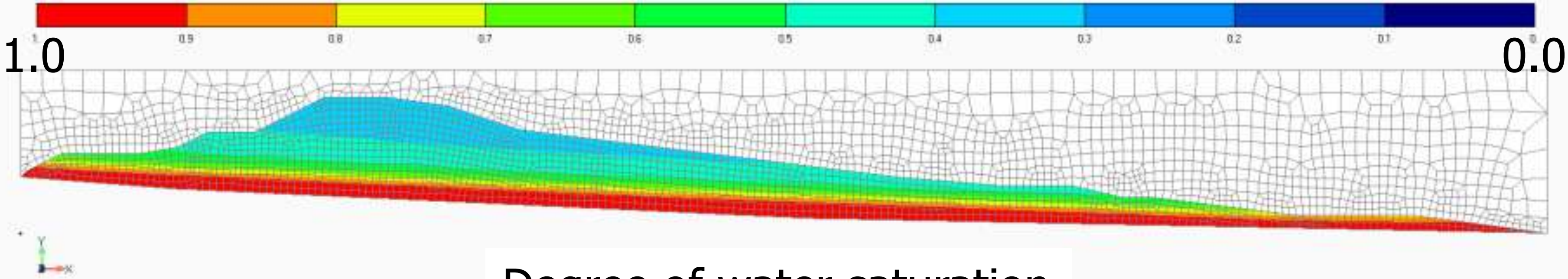


Water level of dammed lake

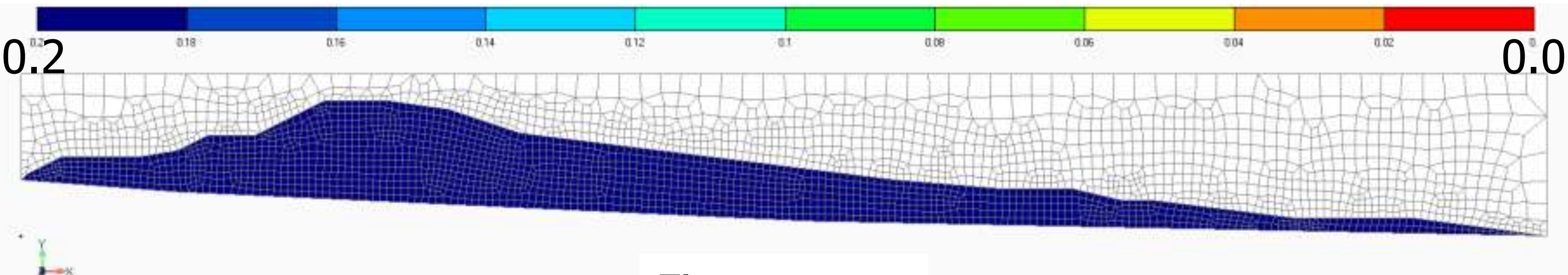


2-D Finite element model of cross section along the reiver

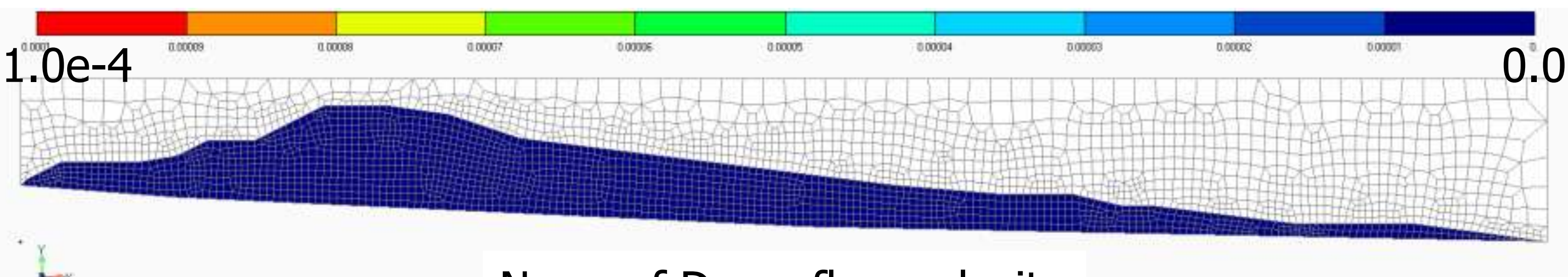
Initial condition



Degree of water saturation

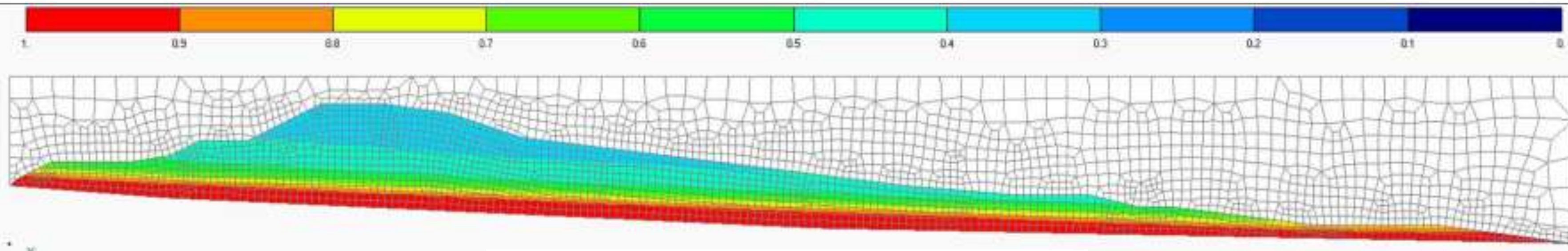


Fine content

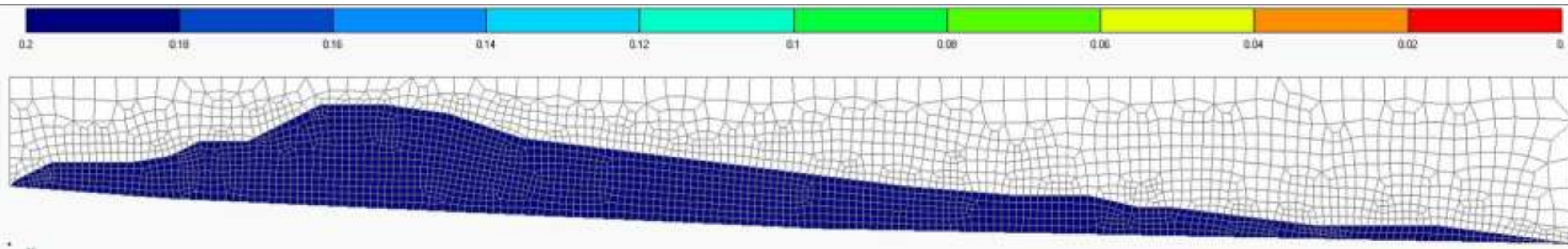


Norm of Darcy flow velocity

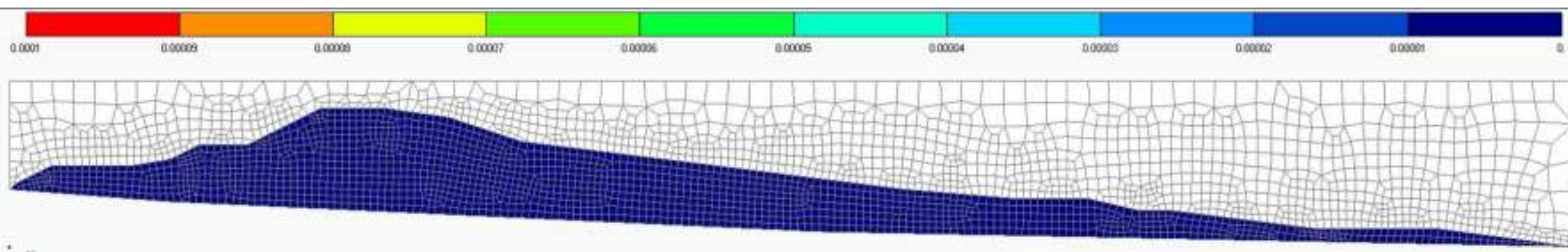
Formation of water flow path in landslide dam



Degree of water saturation

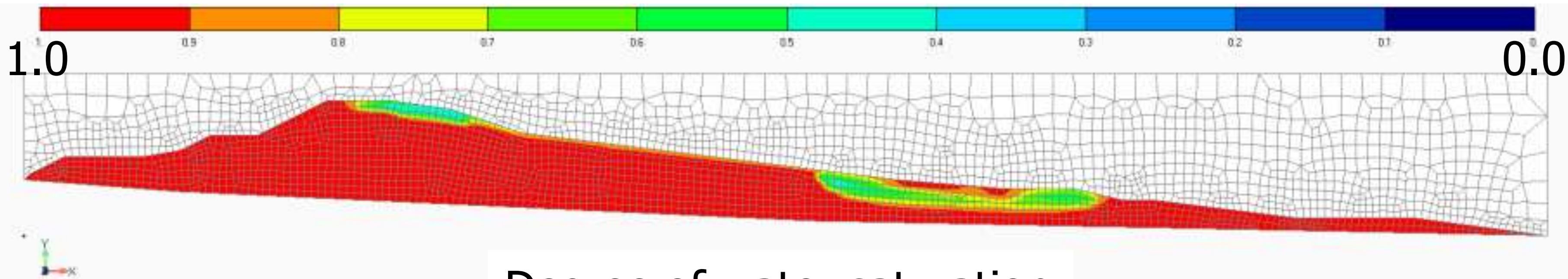


Fine content

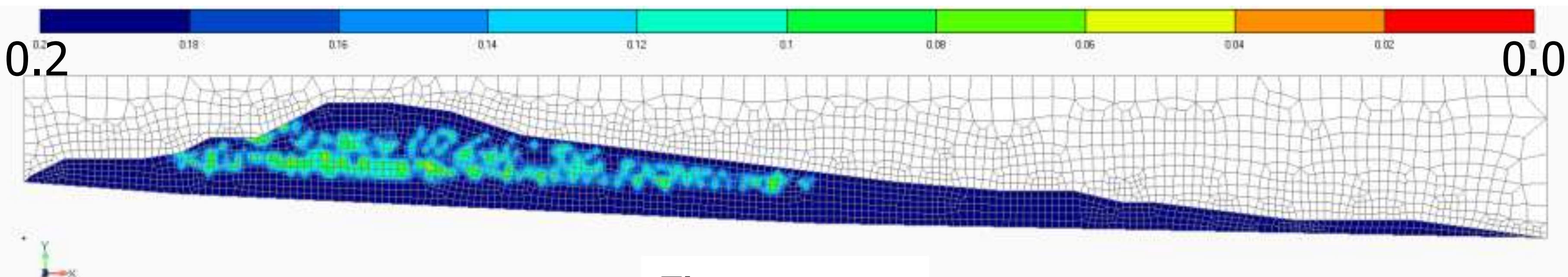


Norm of Darcy flow velocity

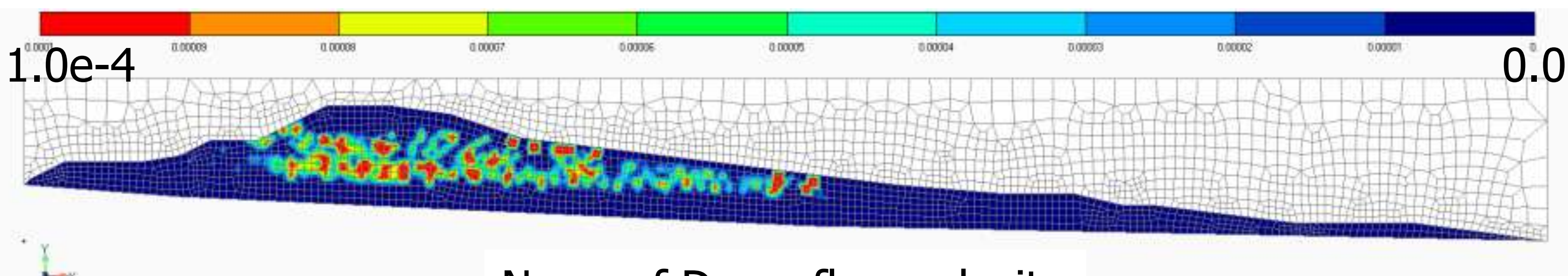
After 500 hours



Degree of water saturation



Fine content



Norm of Darcy flow velocity

Conclusions

- Hydro-mechanical analysis for internal erosion based on porous media theory is presented. The internal erosion of soil skeleton and transport of fine particles in pore water are modeled by mass exchange between soil skeleton and pore water. A rate of mass exchange is assumed to be proportional to the material time derivative of fine contents based on the past laboratory tests.
- In the first validation example, the numerical model reproduced the time histories of eroded mass rate with different hydraulic gradients obtained by the laboratory internal erosion tests.
- In the second simulation example, decrease in fine content and increase in Darcy flow velocity occurred in the landslide dam, which means the formation of water flow path in the landslide dam.

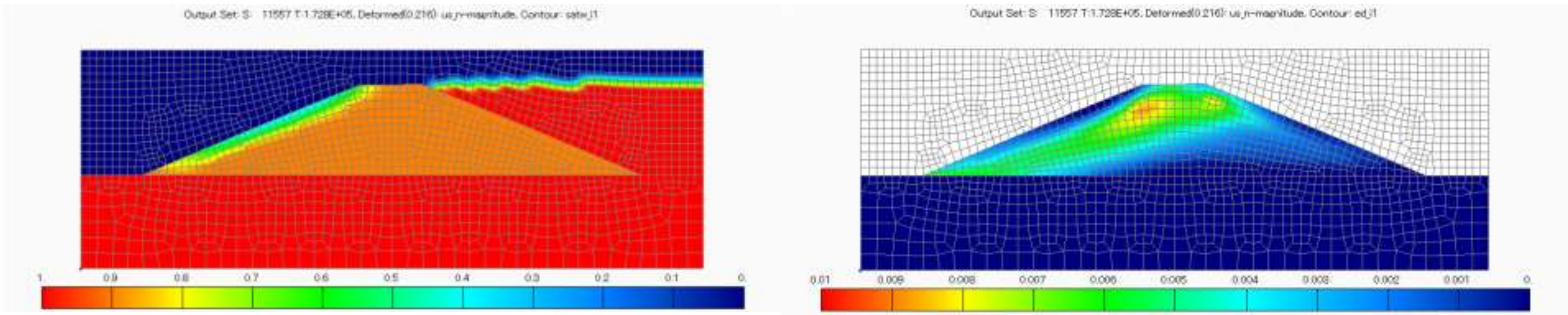
Appendix

Recent studies on numerical analysis

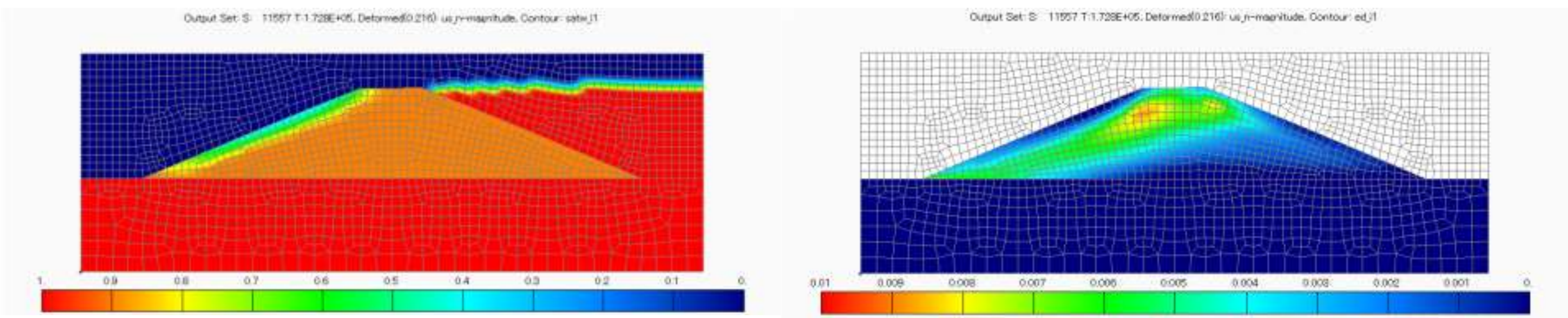
- Surface erosion
  - Formation and migration of headcuts (Wang and Bowles 2007)
  - Parameter sensitivity of discharge through breach (Faeh 2007)
- Internal erosion
  - Seepage-induced erosion and settlement (Cividini et al. 2009)
  - Erosion and transport of fine particle (Fujisawa et al. 2010)
  - Combination of pipe flow in the erosion channel and seepage analysis (Zhou 2012)
- Seepage/overtopping and deformation
  - Large deformation at the toe due to large hydraulic gradient
  - Slip-like strain localization along the slope due to long overflow (Oka et al. 2010)
  - Different distributions of deviatoric strain with and without overtopping (Uzuoka et al. 2011)

Seepage/overtopping and deformation

- Different distributions of deviatoric strain with and without overtopping



Only seepage after 48 hours



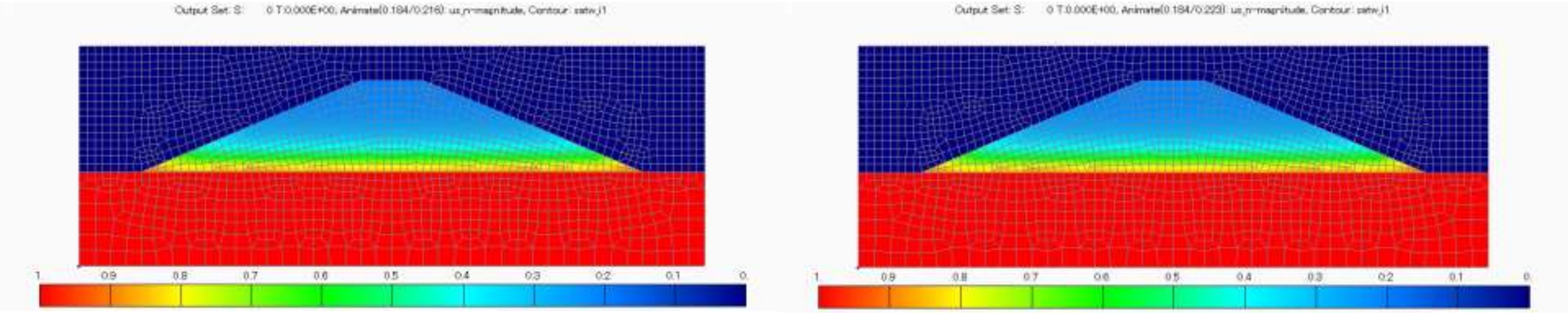
Seepage with overtopping after 27 hours

Degree of saturation

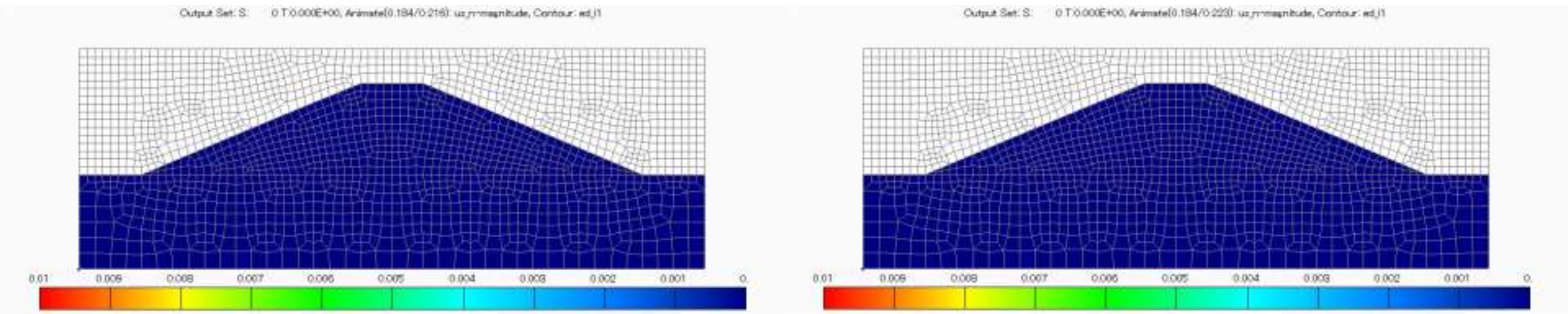
Deviatoric strain

Seepage and overtopping

- Deviatoric strain dependent on saturation



Degree of saturation

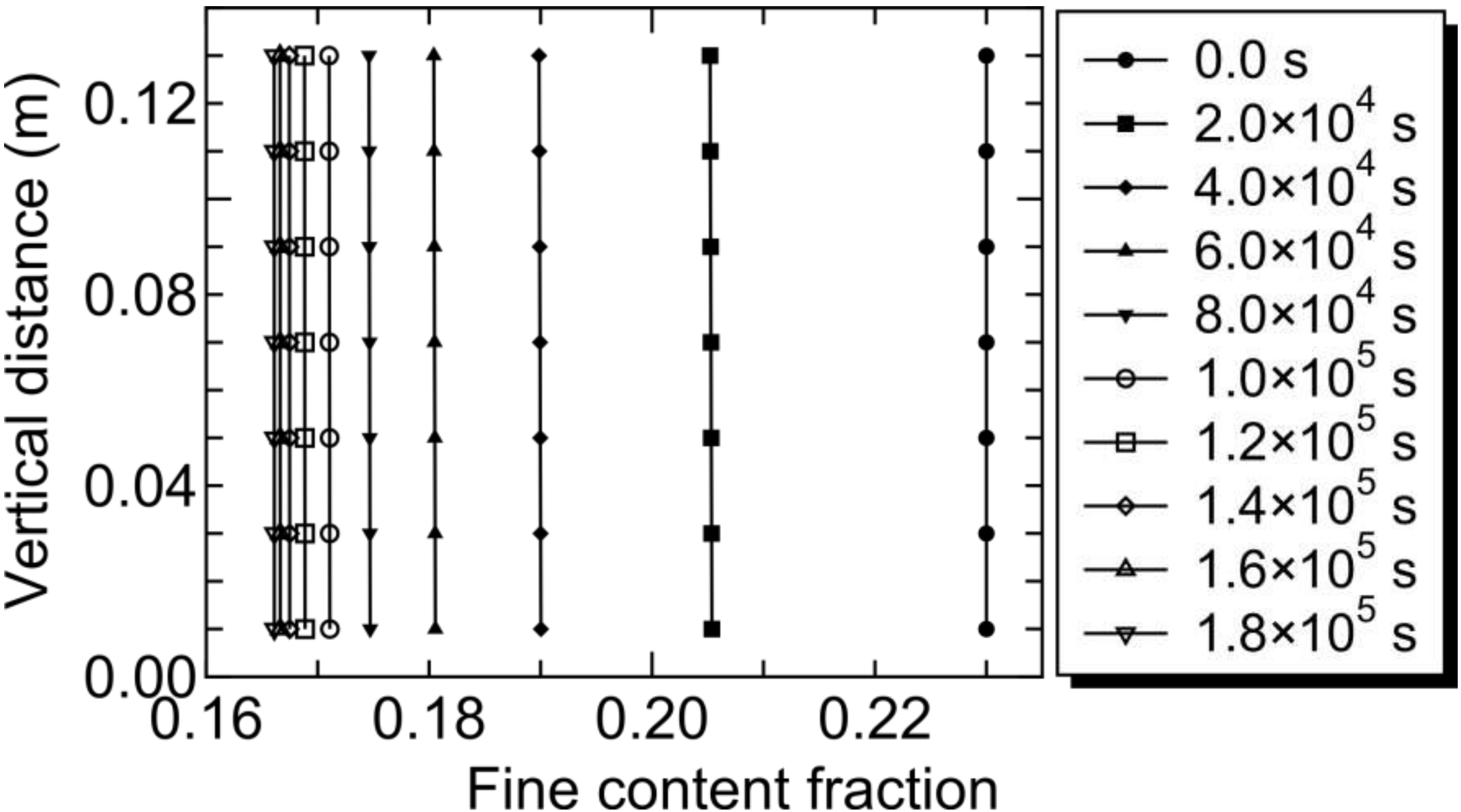


Deviatoric strain

Seepage only | Seepage and overtopping

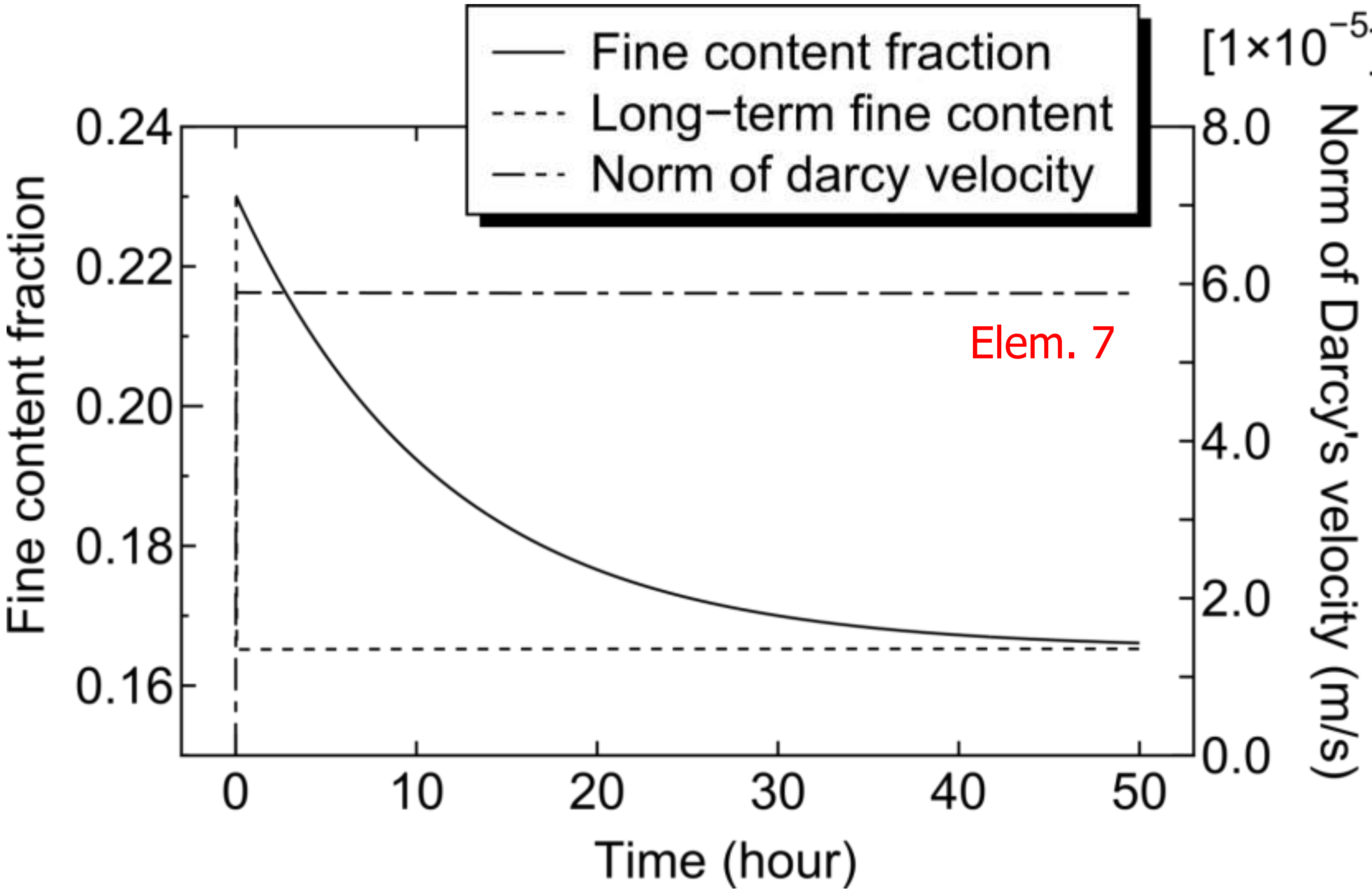
Validation of erosion model

- Internal erosion test (Sterpi 2003)



Validation of erosion model

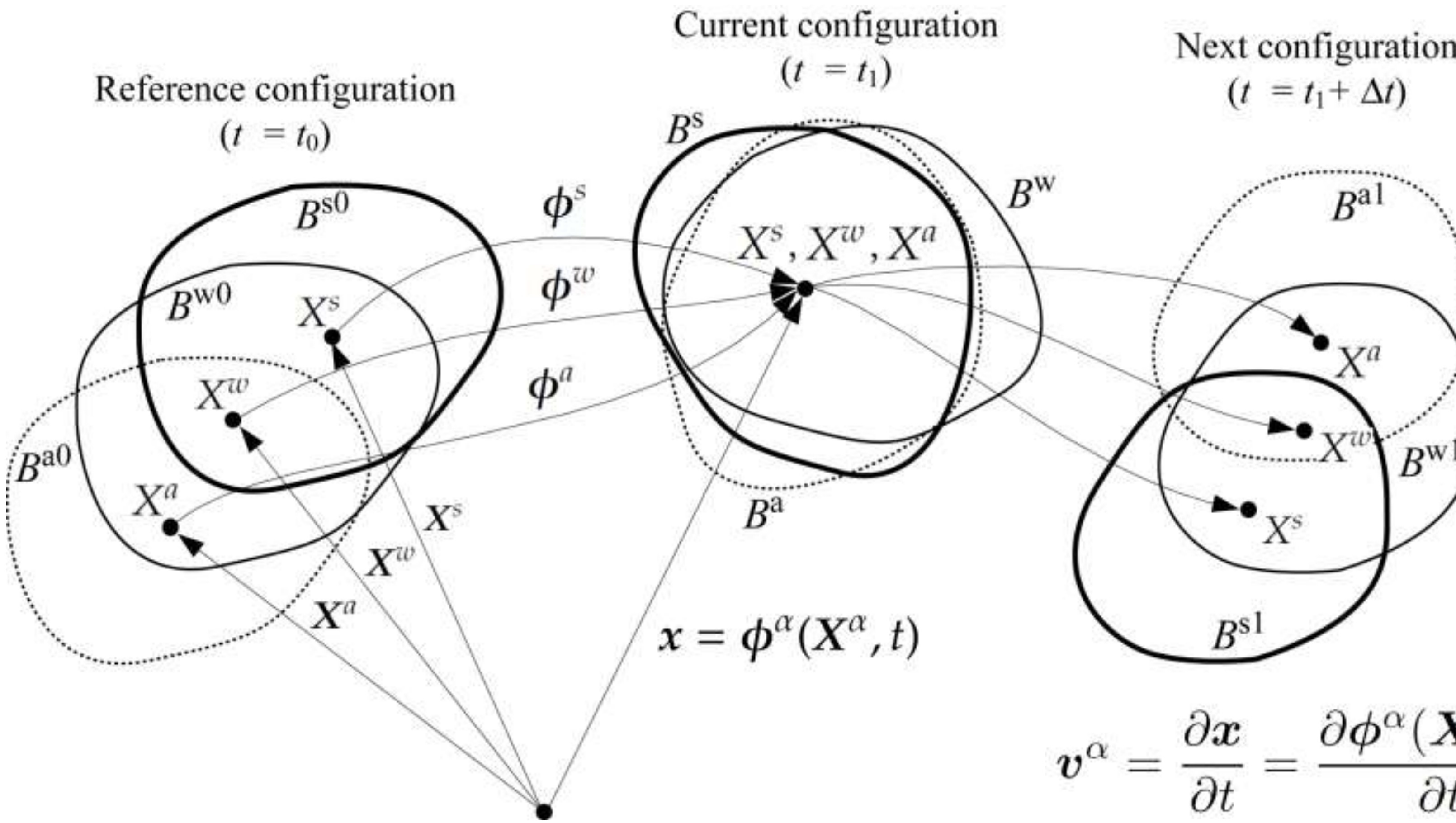
- Internal erosion test (Sterpi 2003)



Formulation

Basic assumptions

- The spatial point with each material *X* in the current configuration is the common point *x*.
- At the common spatial point *x* the motions of the individual phases define the interaction among these phases.

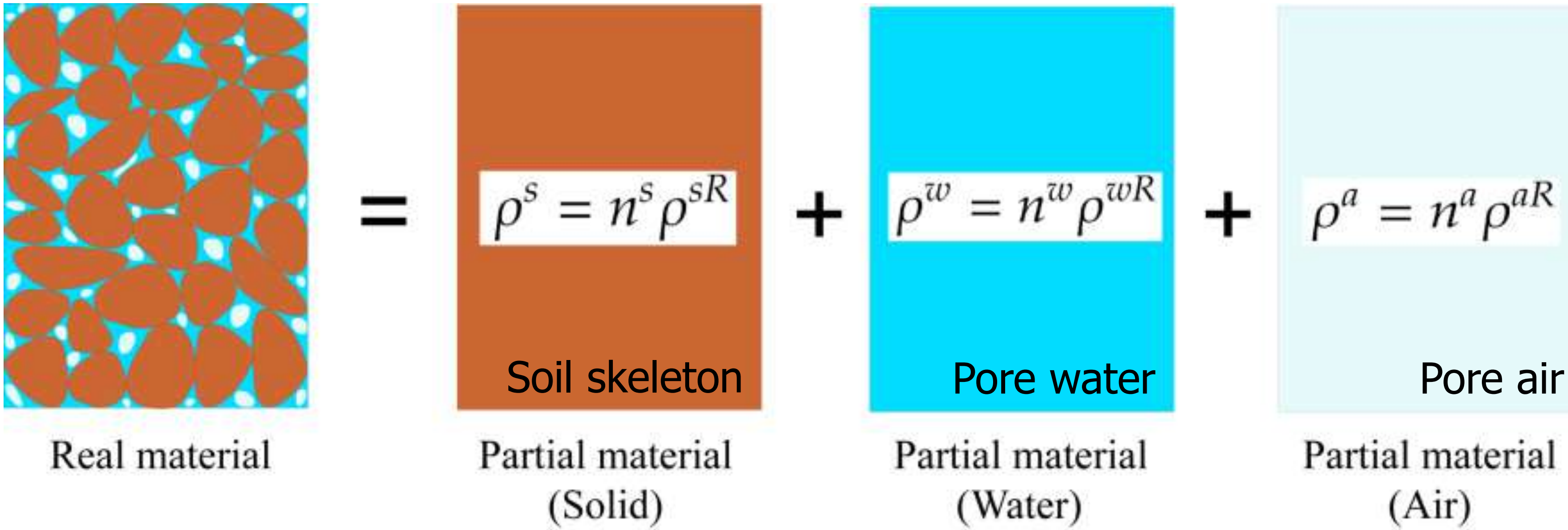


$$v^\alpha = \frac{\partial x}{\partial t} = \frac{\partial \phi^\alpha(X^\alpha, t)}{\partial t}$$

Additional assumptions

1. Isothermal condition,
2. Incompressible soil particle,
3. No mass exchange among phases,
4. Material time derivatives of relative velocity of the pore fluids to the soil skeleton are small compared to the acceleration of the soil skeleton,
5. Advection terms of the pore fluids to the soil skeleton are small compared to the acceleration of the soil skeleton.

Volume fraction concept



Partial density of solid $\rho^{\mathrm{s}} = n^{\mathrm{s}} \rho^{\mathrm{sR}} = \boxed{(1-n)} \rho^{\mathrm{sR}}$

Partial density of water $\rho^{\mathrm{w}} = n^{\mathrm{w}} \rho^{\mathrm{wR}} = \boxed{n s^{\mathrm{w}}} \rho^{\mathrm{wR}}$

Partial density of air $\rho^{\mathrm{a}} = n^{\mathrm{a}} \rho^{\mathrm{aR}} = n s^{\mathrm{a}} \rho^{\mathrm{aR}} = \boxed{n(1-s^{\mathrm{w}})} \rho^{\mathrm{aR}}$

n : porosity

s^{w} : water saturation

Overall density of mixture = superposition of each partial density

$$\rho = \rho^{\mathrm{s}} + \rho^{\mathrm{w}} + \rho^{\mathrm{a}} = (1-n)\rho^{\mathrm{sR}} + n\left(s^{\mathrm{w}} \rho^{\mathrm{wR}} + s^{\mathrm{a}} \rho^{\mathrm{aR}}\right)$$

Description of kinematics

- Based on material time derivative (D^s/Dt) with soil skeleton

$$\boldsymbol{a}^{\mathrm{s}} = \frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{s}}}{Dt} = \frac{\partial \boldsymbol{v}^{\mathrm{s}}}{\partial t} + (\mathrm{grad}\, \boldsymbol{v}^{\mathrm{s}})\boldsymbol{v}^{\mathrm{s}}$$

$$\boldsymbol{a}^{\mathrm{w}} = \frac{D^{\mathrm{w}}\boldsymbol{v}^{\mathrm{w}}}{Dt} = \frac{\partial \boldsymbol{v}^{\mathrm{w}}}{\partial t} + (\mathrm{grad}\, \boldsymbol{v}^{\mathrm{w}})\boldsymbol{v}^{\mathrm{w}}$$

$$= \boldsymbol{a}^{\mathrm{s}} + \frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{ws}}}{Dt} + \{\mathrm{grad}\,(\boldsymbol{v}^{s} + \boldsymbol{v}^{\mathrm{ws}})\}\boldsymbol{v}^{\mathrm{ws}}$$

$$\boldsymbol{a}^{\mathrm{a}} = \frac{D^{\mathrm{a}}\boldsymbol{v}^{\mathrm{a}}}{Dt} = \frac{\partial \boldsymbol{v}^{\mathrm{a}}}{\partial t} + (\mathrm{grad}\, \boldsymbol{v}^{\mathrm{a}})\boldsymbol{v}^{\mathrm{a}}$$

$$= \boldsymbol{a}^{\mathrm{s}} + \frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{as}}}{Dt} + \{\mathrm{grad}\,(\boldsymbol{v}^{s} + \boldsymbol{v}^{\mathrm{as}})\}\boldsymbol{v}^{\mathrm{as}}$$

$\boldsymbol{v}^{\mathrm{ws}}$ $\boldsymbol{v}^{\mathrm{as}}$: relative velocity of pore water and air (Eulerian velocity)

Balance laws for partial materials

- Mass balance (local form)

$$\frac{\partial \rho^\alpha}{\partial t} + \operatorname{div}(\rho^\alpha \boldsymbol{v}^\alpha) = \hat{\rho}^\alpha$$

$\hat{\rho}^\alpha$: mass exchange (neglected here)

- Linear momentum balance (local form)

$$\rho^\alpha \boldsymbol{a}^\alpha = \operatorname{div} \boldsymbol{\sigma}^\alpha + \rho^\alpha \boldsymbol{b}^\alpha + \hat{\boldsymbol{p}}^\alpha$$

$\boldsymbol{\sigma}^\alpha$: Cauchy partial (averaged) total stress tensor of α

$\boldsymbol{b}^\alpha$: body force vector of α

$\hat{\boldsymbol{p}}^\alpha$: interaction force vector of α

$$\sum_{\alpha=1}^{3} (\hat{\rho}^\alpha \boldsymbol{v}^\alpha + \hat{\boldsymbol{p}}^\alpha) = \boldsymbol{0}$$

Recent studies on constitutive model (Gens, 2010)

- First constitutive variables $(\sigma_t)_{ij} - p_g\delta_{ij} + \mu_1(s, S_l)\delta_{ij}$
  - Alonso et al. (1990); Cui & Delage (1996); Vaunat et al. (2000b); Rampino et al. (2000); Chiu & Ng (2003); Georgiadis et al. (2003); Thu et al. (2007); Sheng et al. (2008a)

$$(\sigma_t)_{ij} - p_g\delta_{ij}; \quad \mu_1 = 0$$

Net stress

  - Kohgo et al. (1993); Loret & Khalili (2000, 2002); Laloui et al. (2001); Sun et al. (2003); Russell&Khalili (2006);Masin & Khalili (2008)

$$(\sigma_t)_{ij} - p_g\delta_{ij} + \mu_1(s)\delta_{ij}$$

  - Jommi (2000); Wheeler et al. (2003); Gallipoli et al. (2003); Sheng et al. (2004); Tamagnini (2004); Pereira et al. (2005); Oka et al. (2006); Santagiuliana & Schrefler (2006); Sun et al. (2007a, 2007b); Kohler & Hofstetter (2008); Buscarnera & Nova (2009)

$$(\sigma_t)_{ij} - p_g\delta_{ij} + \mu_1(s, S_l)\delta_{ij}$$

From thermodynamic studies (Hassanizadeh and Gray 1980; Ehlers 1993; Houlsby 1997; Muraleetharan and Wei 1999; Borja 2006; Li 2007)

$$\sigma'_{ij} = (\sigma_t)_{ij} - p_g\delta_{ij} + S_l(p_g - p_l)\delta_{ij}$$

Bishop's stress, average skeleton stress
= compatible with three-phase formulation

- Second constitutive variables
  - Suction, Degree of saturation

$$\mu_2(s, S_l)$$

Constitutive equations (1)

- Effective stress for saturated soil (Terzaghi)
- Effective stress for unsaturated soil

Hassanizadeh and Gray (1980, 1990) , Ehlers (1993), Houlsby (1997), Muraleetharan and Wei (1999), Borja (2004, 2006), Li (2007)

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{s}} + \boldsymbol{\sigma}^{\mathrm{w}} + \boldsymbol{\sigma}^{\mathrm{a}}$$ Total stress = Σ Partial stress

$$= \boldsymbol{\sigma}' - \underline{(s^{\mathrm{w}} p^{\mathrm{w}} + s^{\mathrm{a}} p^{\mathrm{a}})} \boldsymbol{I}$$ Averaged pore pressure

$$= \boldsymbol{\sigma}' - p^{\mathrm{a}} \boldsymbol{I} + p^{\mathrm{a}} \boldsymbol{I} - s^{\mathrm{w}} p^{\mathrm{w}} \boldsymbol{I} - (1 - s^{\mathrm{w}}) p^{\mathrm{a}} \boldsymbol{I}$$

$$= \boldsymbol{\sigma}' - p^{\mathrm{a}} \boldsymbol{I} + s^{\mathrm{w}} (p^{\mathrm{a}} - p^{\mathrm{w}}) \boldsymbol{I}$$

$$(-\boldsymbol{\sigma}') = \underbrace{(-\boldsymbol{\sigma}) - p^{\mathrm{a}} \boldsymbol{I}}_{\text{Net stress}} + \underbrace{s^{\mathrm{w}} (p^{\mathrm{a}} - p^{\mathrm{w}})}_{\text{Suction}} \boldsymbol{I}$$

Net stress Suction $p^{\mathrm{c}} = p^{\mathrm{a}} - p^{\mathrm{w}}$

Ref. Bishop's effective stress

$$(-\boldsymbol{\sigma}') = (-\boldsymbol{\sigma}) - p^{\mathrm{a}} \boldsymbol{I} + \underline{\chi} (p^{\mathrm{a}} - p^{\mathrm{w}}) \boldsymbol{I}$$

Constitutive equations (2)

- Cauchy partial stress

$$\boldsymbol{\sigma}^{\mathrm{s}} = \boldsymbol{\sigma}' - (1-n)(s^{\mathrm{w}}p^{\mathrm{w}} + s^{\mathrm{a}}p^{\mathrm{a}})\boldsymbol{I} = \boldsymbol{\sigma}' - (1-n)p^{\mathrm{f}}\boldsymbol{I}$$

$$\boldsymbol{\sigma}^{\mathrm{w}} = -ns^{\mathrm{w}}p^{\mathrm{w}}\boldsymbol{I}$$

$$\boldsymbol{\sigma}^{\mathrm{a}} = -ns^{\mathrm{a}}p^{\mathrm{a}}\boldsymbol{I}$$

$\boldsymbol{\sigma}'$: Cauchy effective stress tensor

p^{w} : pore water pressure (positive in compression)

p^{a} : pore air pressure (positive in compression)

p^{f} : averaged pore pressure (positive in compression)

Constitutive equations (3)

- Interaction force

Hassanizadeh and Gray (1980, 1990) ,
de Boer (2000) , Schrefler (2002)

$$\hat{\boldsymbol{p}}^{\mathrm{s}} = -\hat{\boldsymbol{p}}^{\mathrm{w}} - \hat{\boldsymbol{p}}^{\mathrm{a}}$$

$$\begin{aligned}\hat{\boldsymbol{p}}^{\mathrm{w}} &= p^{\mathrm{w}} \operatorname{grad} n^{\mathrm{w}} - \boldsymbol{\mu}^{\mathrm{w}} n^{\mathrm{w}} \boldsymbol{v}^{\mathrm{ws}} \\ &= p^{\mathrm{w}} \operatorname{grad} (n s^{\mathrm{w}}) - \boldsymbol{\mu}^{\mathrm{w}} n s^{\mathrm{w}} \boldsymbol{v}^{\mathrm{ws}}\end{aligned}$$

$$\begin{aligned}\hat{\boldsymbol{p}}^{\mathrm{a}} &= p^{\mathrm{a}} \operatorname{grad} n^{\mathrm{a}} - \boldsymbol{\mu}^{\mathrm{a}} n^{\mathrm{a}} \boldsymbol{v}^{\mathrm{as}} \\ &= p^{\mathrm{a}} \operatorname{grad} (n s^{\mathrm{a}}) - \boldsymbol{\mu}^{\mathrm{a}} n s^{\mathrm{a}} \boldsymbol{v}^{\mathrm{as}}\end{aligned}$$

with Darcy’s law

$$\boldsymbol{\mu}^{\mathrm{w}} = \frac{n^{\mathrm{w}} \rho^{\mathrm{wR}} g}{\boldsymbol{k}^{\mathrm{ws}}} = \frac{n s^{\mathrm{w}} \rho^{\mathrm{wR}} g}{k^{\mathrm{ws}}} \boldsymbol{I}$$

k^{ws} : permeability coefficient of water

$$\boldsymbol{\mu}^{\mathrm{a}} = \frac{n^{\mathrm{a}} \rho^{\mathrm{aR}} g}{\boldsymbol{k}^{\mathrm{as}}} = \frac{n s^{\mathrm{a}} \rho^{\mathrm{aR}} g}{k^{\mathrm{as}}} \boldsymbol{I}$$

k^{as} : permeability coefficient of air

Constitutive equations (4)

- Hyperelastic material for soil skeleton (neo-Hookean model)

$$\boldsymbol{\sigma}' = \frac{\mu_{\mathrm{n}}}{J^{\mathrm{s}}}(\boldsymbol{b}^{\mathrm{s}} - \boldsymbol{I}) + \frac{\lambda_{\mathrm{n}}}{J^{\mathrm{s}}}(\ln J^{\mathrm{s}})\boldsymbol{I}$$

(incompressible soil particle)

$\lambda, \ \mu$: Lame's constants $\quad$ J^{s} : Jacobian for soil skeleton

$\boldsymbol{b}^{\mathrm{s}}$: left Cauchy-Green deformation tensor for soil skeleton

- State equation for water (isothermal condition)

$$\frac{D^{\mathrm{s}}\rho^{\mathrm{wR}}}{Dt} = \frac{\rho^{\mathrm{wR}}}{(K^{\mathrm{w}})_s}\frac{D^{\mathrm{s}}p^{\mathrm{w}}}{Dt}$$

K^{w} : bulk modulus for water

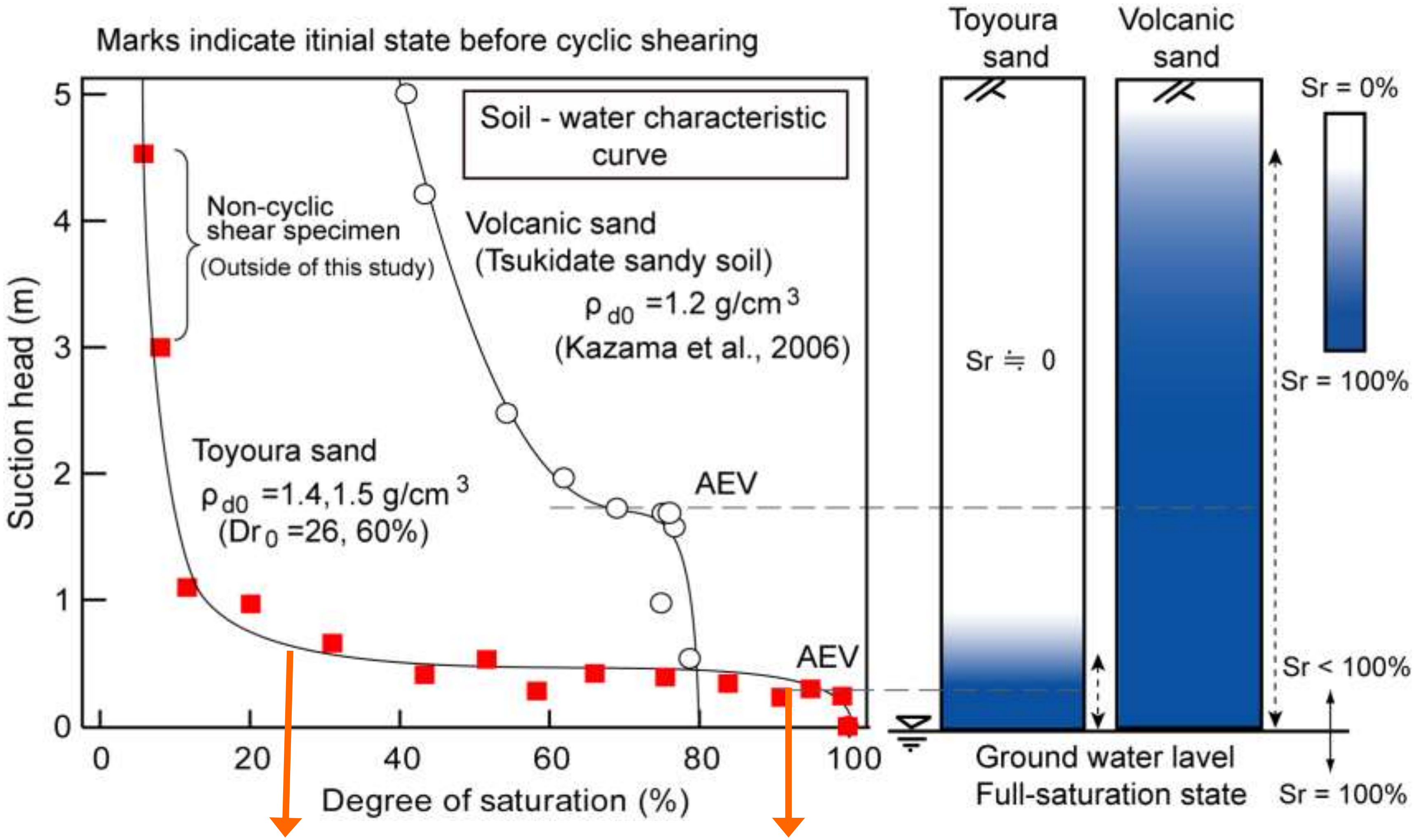
- State equation for air (isothermal condition)

$$\frac{D^{\mathrm{s}}p^{\mathrm{a}}}{Dt} = \Theta\bar{R}\frac{D^{\mathrm{s}}\rho^{\mathrm{aR}}}{Dt}$$

Θ : absolute temperature

$\bar{R}$: specific gas constant

Water retention curve (WRC, SWCC)

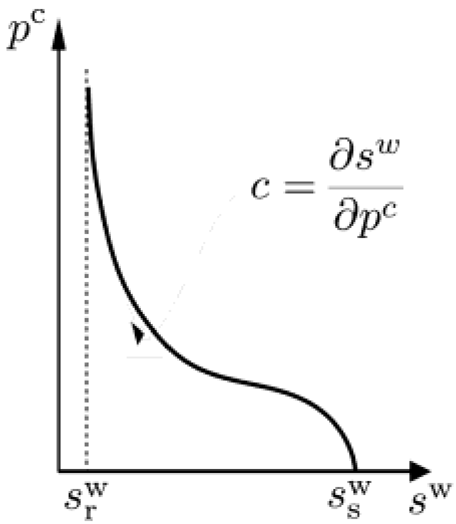


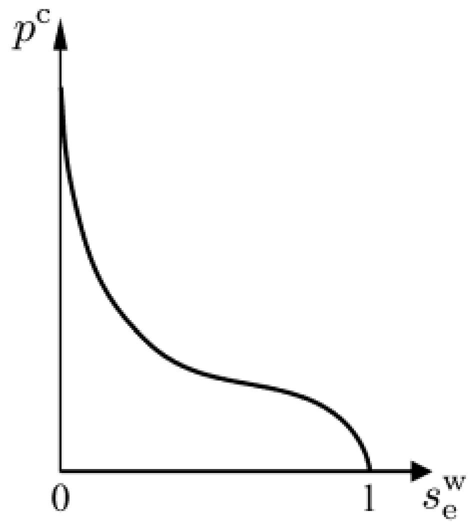
Low saturation:
Large suction and effective stress
(under constant net stress)

High saturation:
Small suction and effective stress
(under constant net stress)

Constitutive equations (5)

- SWCC (van Genuchten, 1980）





$$s^{w} = s^{w}(p^{c}, T) = s^{w}(p^{c})$$

(isothermal condition)

$$\frac{D^{s} s^{w}}{Dt} = \frac{\partial s^{w}}{\partial p^{c}} \frac{D^{s} p^{c}}{Dt} = c \frac{D^{s} p^{c}}{Dt}$$

$$s_{e}^{w}(p^{c}) = \{1 + (\alpha_{vg} p^{c})^{n_{vg}}\}^{-m_{vg}}$$

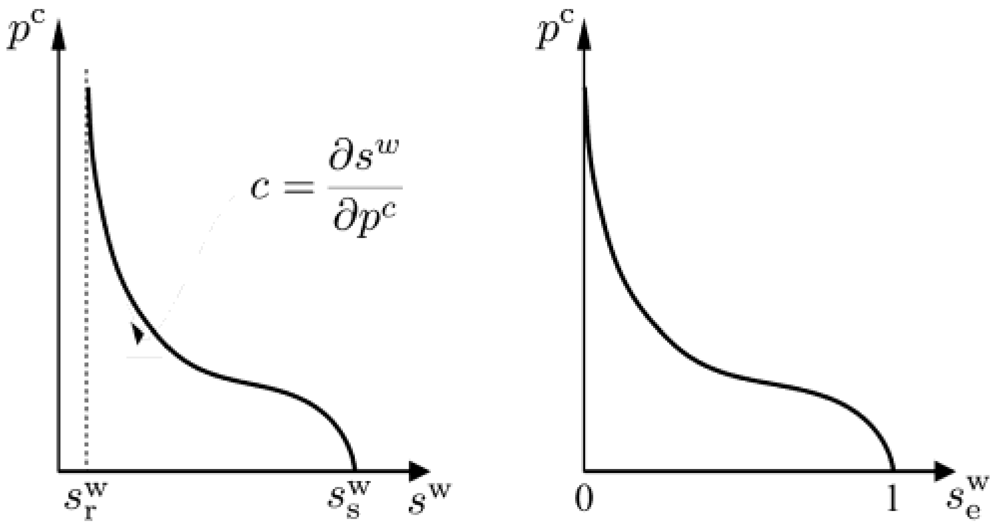
$$s_{e}^{w}(p^{c}) = \frac{\theta - \theta_{r}}{\theta_{s} - \theta_{r}} = \frac{s^{w} - s_{r}^{w}}{s_{s}^{w} - s_{r}^{w}}$$

$$k^{ws} = k_{s}^{ws} \{s_{e}^{w}\}^{\xi_{vg}} \left[1 - \left\{1 - (s_{e}^{w})^{\frac{1}{m_{vg}}}\right\}^{m_{vg}}\right]^{2} = k_{s}^{ws} k_{r}^{ws}$$

$$k^{as} = k_{s}^{as} (1 - s_{e}^{w})^{\eta_{vg}} \left\{1 - (s_{e}^{w})^{\frac{1}{m_{vg}}}\right\}^{2m_{vg}} = k_{s}^{as} k_{r}^{as}$$

Soil water characteristic curve

- Continuous logistic function



$$s^{\mathrm{w}} = s^{\mathrm{w}}(p^{\mathrm{c}}, T) = s^{\mathrm{w}}(p^{\mathrm{c}})$$

(isothermal condition)

$$\frac{D^{\mathrm{s}} s^{\mathrm{w}}}{Dt} = \frac{\partial s^{\mathrm{w}}}{\partial p^{\mathrm{c}}} \frac{D^{\mathrm{s}} p^{\mathrm{c}}}{Dt} = c \frac{D^{\mathrm{s}} p^{\mathrm{c}}}{Dt}$$

$$s_e^{\mathrm{w}}(p^{\mathrm{c}}) = \frac{1}{\{1 + \exp(a_{\mathrm{lg}} p^{\mathrm{c}} + b_{\mathrm{lg}})\}^{c_{\mathrm{lg}}}}$$

$$s_{\mathrm{e}}^{\mathrm{w}}(p^{\mathrm{c}}) = \frac{\theta - \theta_{\mathrm{r}}}{\theta_{\mathrm{s}} - \theta_{\mathrm{r}}} = \frac{s^{\mathrm{w}} - s_{\mathrm{r}}^{\mathrm{w}}}{s_{\mathrm{s}}^{\mathrm{w}} - s_{\mathrm{r}}^{\mathrm{w}}}$$

$$k^{\mathrm{ws}} = k_{\mathrm{s}}^{\mathrm{ws}} (s_{\mathrm{e}}^{\mathrm{w}})^{\xi_{\mathrm{vg}}}$$

$$k^{\mathrm{as}} = k_{\mathrm{s}}^{\mathrm{as}} (1 - s_{\mathrm{e}}^{\mathrm{w}})^{\eta_{\mathrm{vg}}}$$

Constitutive equations (6)

- Extended Modified Cam-Clay model

Yield function

$$\bar{F} = \frac{Q^2}{M^2} + P' \left(P' - \bar{P}'_{\mathrm{c}}\right) = \frac{Q^2}{M^2} + P' \left(P' - \frac{1}{R^*}\tilde{P}'_{\mathrm{c}}\right)$$

Evolution law

$$\dot{\tilde{P}}'_{\mathrm{c}} = \tilde{U}\,\dot{\epsilon}^{\mathrm{p}}_{\mathrm{v}} = -\frac{1}{\hat{\lambda} - \hat{\kappa}}\tilde{P}'_{\mathrm{c}}\,\dot{\epsilon}^{\mathrm{p}}_{\mathrm{v}} = -\hat{\theta}\tilde{P}'_{\mathrm{c}}\,\dot{\epsilon}^{\mathrm{p}}_{\mathrm{v}}$$

$$R^* = \exp\left\{-a^{\mathrm{c}}\left(\frac{s^{\mathrm{w}}\theta^{\mathrm{c}}}{p^*}\right)^{b^{\mathrm{c}}}\right\} \quad (\theta^{\mathrm{c}} \geq 0)$$

$$= 1 \quad (\theta^{\mathrm{c}} < 0)$$

Constitutive equations (7)

- Extended Modified Cam-Clay model

Pressure-dependent hyperelastic model (Borja and Tamagnini 1998)

$$\phi\left(\epsilon_{\mathrm{v}}^{\mathrm{e}}, \epsilon_{\mathrm{d}}^{\mathrm{e}}\right) = \tilde{\phi}\left(\epsilon_{\mathrm{v}}^{\mathrm{e}}\right) + \frac{3}{2}\mu^{\mathrm{e}}\left(\epsilon_{\mathrm{d}}^{\mathrm{e}}\right)^2$$

$$\tilde{\phi}\left(\epsilon_{\mathrm{v}}^{\mathrm{e}}\right) = -P_0'\hat{\kappa}\exp\left(-\frac{\epsilon_{\mathrm{v}}^{\mathrm{e}} - \epsilon_{\mathrm{v0}}^{\mathrm{e}}}{\hat{\kappa}}\right) = -P_0'\hat{\kappa}\exp\Omega$$

$$\mu^{\mathrm{e}} = \mu_0 + \frac{\alpha}{\hat{\kappa}}\tilde{\phi} = \mu_0 - P_0'\alpha\exp\Omega$$

Full formulation (1)

- Linear momentum balance of mixture

$$\rho \boldsymbol{a}^{\mathrm{s}} + \rho^{\mathrm{w}} \left[\frac{D^{\mathrm{s}} \boldsymbol{v}^{\mathrm{ws}}}{Dt} + \left\{ \operatorname{grad} \left(\boldsymbol{v}^{s} + \boldsymbol{v}^{\mathrm{ws}} \right) \right\} \boldsymbol{v}^{\mathrm{ws}} \right]$$

$$+ \rho^{\mathrm{a}} \left[\frac{D^{\mathrm{s}} \boldsymbol{v}^{\mathrm{as}}}{Dt} + \left\{ \operatorname{grad} \left(\boldsymbol{v}^{s} + \boldsymbol{v}^{\mathrm{as}} \right) \right\} \boldsymbol{v}^{\mathrm{as}} \right]$$

$$= \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b}$$

$$\rho \boldsymbol{a}^{\mathrm{s}} = \operatorname{div} \left(\boldsymbol{\sigma}' - p^{\mathrm{f}} \boldsymbol{I} \right) + \rho \boldsymbol{b}$$

Full formulation (2)

- Linear momentum balance of water

$$
ns^{\mathrm{w}}\rho^{\mathrm{wR}}\left[\boldsymbol{a}^{\mathrm{s}} + \frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{ws}}}{Dt} + \left\{\mathrm{grad}\left(\boldsymbol{v}^{s} + \boldsymbol{v}^{\mathrm{ws}}\right)\right\}\boldsymbol{v}^{\mathrm{ws}}\right]
$$

$$
= -ns^{\mathrm{w}}\mathrm{grad}\,p^{\mathrm{w}} + ns^{\mathrm{w}}\rho^{\mathrm{wR}}\boldsymbol{b} - \frac{ns^{\mathrm{w}}\rho^{\mathrm{wR}}g}{\boldsymbol{k}^{\mathrm{ws}}}ns^{\mathrm{w}}\boldsymbol{v}^{\mathrm{ws}}
$$

- Linear momentum balance of air

$$
ns^{\mathrm{a}}\rho^{\mathrm{aR}}\left[\boldsymbol{a}^{\mathrm{s}} + \frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{as}}}{Dt} + \left\{\mathrm{grad}\left(\boldsymbol{v}^{s} + \boldsymbol{v}^{\mathrm{as}}\right)\right\}\boldsymbol{v}^{\mathrm{as}}\right]
$$

$$
= -ns^{\mathrm{a}}\mathrm{grad}\,p^{\mathrm{a}} + ns^{\mathrm{a}}\rho^{\mathrm{aR}}\boldsymbol{b} - \frac{ns^{\mathrm{a}}\rho^{\mathrm{aR}}g}{\boldsymbol{k}^{\mathrm{as}}}ns^{\mathrm{a}}\boldsymbol{v}^{\mathrm{as}}
$$

Full formulation (3)

- Mass balance of water

$$\left(\frac{n s^{\mathrm{w}} \rho^{\mathrm{wR}}}{K^{\mathrm{w}}} - n\rho^{\mathrm{wR}} c\right) \frac{D^{\mathrm{s}} p^{\mathrm{w}}}{Dt} + n\rho^{\mathrm{wR}} c \frac{D^{\mathrm{s}} p^{\mathrm{a}}}{Dt}$$

$$+ s^{\mathrm{w}} \rho^{\mathrm{wR}} \operatorname{div} \boldsymbol{v}^{\mathrm{s}} + \operatorname{div}\left(n s^{\mathrm{w}} \rho^{\mathrm{wR}} \boldsymbol{v}^{\mathrm{ws}}\right) = 0$$

- Mass balance of air

$$\left\{\frac{n(1 - s^{\mathrm{w}})}{\Theta \bar{R}} - n\rho^{\mathrm{aR}} c\right\} \frac{D^{\mathrm{s}} p^{\mathrm{a}}}{Dt} + n\rho^{\mathrm{aR}} c \frac{D^{\mathrm{s}} p^{\mathrm{w}}}{Dt}$$

$$+ (1 - s^{\mathrm{w}})\rho^{\mathrm{aR}} \operatorname{div} \boldsymbol{v}^{\mathrm{s}} + \operatorname{div}\left(n s^{\mathrm{a}} \rho^{\mathrm{aR}} \boldsymbol{v}^{\mathrm{as}}\right) = 0$$

Simplified formulation (1)

- Linear momentum balance of mixture
  - Material time derivatives of relative velocity of the pore fluids to the soil skeleton are small compared to the acceleration of the soil skeleton.
  - Advection terms of the pore fluids to the soil skeleton are small compared to the acceleration of the soil skeleton.

$$
\begin{aligned}
\rho \boldsymbol{a}^{\mathrm{s}} &\cancel{+ \rho^{\mathrm{w}} \left[\frac{D^{\mathrm{s}} \boldsymbol{v}^{\mathrm{ws}}}{Dt} + \left\{ \mathrm{grad} \left(\boldsymbol{v}^{\mathrm{s}} + \boldsymbol{v}^{\mathrm{ws}} \right) \right\} \boldsymbol{v}^{\mathrm{ws}} \right]} \\
&\cancel{+ \rho^{\mathrm{a}} \left[\frac{D^{\mathrm{s}} \boldsymbol{v}^{\mathrm{as}}}{Dt} + \left\{ \mathrm{grad} \left(\boldsymbol{v}^{\mathrm{s}} + \boldsymbol{v}^{\mathrm{as}} \right) \right\} \boldsymbol{v}^{\mathrm{as}} \right]} \\
&= \mathrm{div}\, \boldsymbol{\sigma} + \rho \boldsymbol{b}
\end{aligned}
$$

$$
\rho \boldsymbol{a}^{\mathrm{s}} = \mathrm{div} \left(\boldsymbol{\sigma}' - p^{\mathrm{f}} \boldsymbol{I} \right) + \rho \boldsymbol{b}
$$

Simplified formulation (2)

- Linear momentum balance of water

$$
ns^{\mathrm{w}}\rho^{\mathrm{wR}}\left[\boldsymbol{a}^{\mathrm{s}}+\cancel{\frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{ws}}}{Dt}+\{\operatorname{grad}(\boldsymbol{v}^{\mathrm{s}}+\boldsymbol{v}^{\mathrm{ws}})\}\boldsymbol{v}^{\mathrm{ws}}}\right]
$$

$$
=-ns^{\mathrm{w}}\operatorname{grad}p^{\mathrm{w}}+ns^{\mathrm{w}}\rho^{\mathrm{wR}}\boldsymbol{b}-\frac{ns^{\mathrm{w}}\rho^{\mathrm{wR}}g}{\boldsymbol{k}^{\mathrm{ws}}}ns^{\mathrm{w}}\boldsymbol{v}^{\mathrm{ws}}
$$

- Linear momentum balance of air

$$
ns^{\mathrm{a}}\rho^{\mathrm{aR}}\left[\boldsymbol{a}^{\mathrm{s}}+\cancel{\frac{D^{\mathrm{s}}\boldsymbol{v}^{\mathrm{as}}}{Dt}+\{\operatorname{grad}(\boldsymbol{v}^{s}+\boldsymbol{v}^{\mathrm{as}})\}\boldsymbol{v}^{\mathrm{as}}}\right]
$$

$$
=-ns^{\mathrm{a}}\operatorname{grad}p^{\mathrm{a}}+ns^{\mathrm{a}}\rho^{\mathrm{aR}}\boldsymbol{b}-\frac{ns^{\mathrm{a}}\rho^{\mathrm{aR}}g}{\boldsymbol{k}^{\mathrm{as}}}ns^{\mathrm{a}}\boldsymbol{v}^{\mathrm{as}}
$$

Simplified formulation (3)

- Linear momentum balance of water
  - Generalized Darcy's law

$$n s^{\mathrm{w}} \boldsymbol{v}^{\mathrm{ws}} = \frac{\boldsymbol{k}^{\mathrm{ws}}}{\rho^{\mathrm{wR}} g} \left\{ -\operatorname{grad} p^{\mathrm{w}} + \rho^{\mathrm{wR}} (\boldsymbol{b}^{\mathrm{w}} - \boldsymbol{a}^{\mathrm{s}}) \right\}$$

- Linear momentum balance of air
  - Generalized Darcy's law

$$n s^{\mathrm{a}} \boldsymbol{v}^{\mathrm{as}} = \frac{\boldsymbol{k}^{\mathrm{as}}}{\rho^{\mathrm{aR}} g} \left\{ -\operatorname{grad} p^{\mathrm{a}} + \rho^{\mathrm{aR}} (\boldsymbol{b}^{\mathrm{a}} - \boldsymbol{a}^{\mathrm{s}}) \right\}$$

Simplified formulation (4)

- Mass and momentum balance of water

$$\left(\frac{ns^{\mathrm{w}}\rho^{\mathrm{wR}}}{K^{\mathrm{w}}} - n\rho^{\mathrm{wR}}c\right)\frac{D^{\mathrm{s}}p^{\mathrm{w}}}{Dt} + n\rho^{\mathrm{wR}}c\frac{D^{\mathrm{s}}p^{\mathrm{a}}}{Dt} + s^{\mathrm{w}}\rho^{\mathrm{wR}}\mathrm{div}\,\boldsymbol{v}^{\mathrm{s}}$$

$$+\,\mathrm{div}\left\{\frac{\boldsymbol{k}^{\mathrm{ws}}}{g}(-\mathrm{grad}\,p^{\mathrm{w}} + \rho^{\mathrm{wR}}\boldsymbol{b}^{\mathrm{w}} - \rho^{\mathrm{wR}}\boldsymbol{a}^{\mathrm{s}})\right\} = 0$$

- Mass and momentum balance of air

$$\left\{\frac{n(1-s^{\mathrm{w}})}{\Theta\bar{R}} - n\rho^{\mathrm{aR}}c\right\}\frac{D^{\mathrm{s}}p^{\mathrm{a}}}{Dt} + n\rho^{\mathrm{aR}}c\frac{D^{\mathrm{s}}p^{\mathrm{w}}}{Dt} + (1-s^{\mathrm{w}})\rho^{\mathrm{aR}}\mathrm{div}\,\boldsymbol{v}^{\mathrm{s}}$$

$$+\,\mathrm{div}\left\{\frac{\boldsymbol{k}^{\mathrm{as}}}{g}(-\mathrm{grad}\,p^{\mathrm{a}} + \rho^{\mathrm{aR}}\boldsymbol{b}^{\mathrm{a}} - \rho^{\mathrm{aR}}\boldsymbol{a}^{\mathrm{s}})\right\} = 0$$

Simplified governing equations

- Linear momentum balance of mixture

$$\rho \boldsymbol{a}^{\mathrm{s}} = \operatorname{div} \boldsymbol{\sigma} + \rho \boldsymbol{b}$$

$$= \operatorname{div} (\boldsymbol{\sigma}' - p^{\mathrm{f}} \boldsymbol{I}) + \rho \boldsymbol{b}$$

- Mass and linear momentum balance of water

$$\left(\frac{n s^{\mathrm{w}} \rho^{\mathrm{wR}}}{K^{\mathrm{w}}} - n \rho^{\mathrm{wR}} c \right) \frac{D^{\mathrm{s}} p^{\mathrm{w}}}{Dt} + n \rho^{\mathrm{wR}} c \frac{D^{\mathrm{s}} p^{\mathrm{a}}}{Dt} + s^{\mathrm{w}} \rho^{\mathrm{wR}} \operatorname{div} \boldsymbol{v}^{\mathrm{s}}$$

$$+ \operatorname{div} \left\{ \frac{\boldsymbol{k}^{\mathrm{ws}}}{g} (-\operatorname{grad} p^{\mathrm{w}} + \rho^{\mathrm{wR}} \boldsymbol{b}^{\mathrm{w}} - \rho^{\mathrm{wR}} \boldsymbol{a}^{\mathrm{s}}) \right\} = 0$$

- M

$$\left\{ \frac{n(1 - s^{\mathrm{w}})}{\Theta \bar{R}} - n \rho^{\mathrm{aR}} c \right\} \frac{D^{\mathrm{s}} p^{\mathrm{a}}}{Dt} + n \rho^{\mathrm{aR}} c \frac{D^{\mathrm{s}} p^{\mathrm{w}}}{Dt} + (1 - s^{\mathrm{w}}) \rho^{\mathrm{aR}} \operatorname{div} \boldsymbol{v}^{\mathrm{s}}$$

$$+ \operatorname{div} \left\{ \frac{\boldsymbol{k}^{\mathrm{as}}}{g} (-\operatorname{grad} p^{\mathrm{a}} + \rho^{\mathrm{aR}} \boldsymbol{b}^{\mathrm{a}} - \rho^{\mathrm{aR}} \boldsymbol{a}^{\mathrm{s}}) \right\} = 0$$

Weak forms (1)

- Linear momentum balance of mixture

$$
\begin{aligned}
\delta w^{\mathrm{s}} &= \delta w^{\mathrm{s}}_{(1)} + \delta w^{\mathrm{s}}_{(2)} + \delta w^{\mathrm{s}}_{(3)} + \delta w^{\mathrm{s}}_{(4)} + \delta w^{\mathrm{s}}_{(5)} \\
\delta w^{\mathrm{s}}_{(1)} &= \int_{B^{\mathrm{s}}} \rho \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} dv = \delta w^{\mathrm{s}}_{(1)}(\underline{\boldsymbol{a}^{\mathrm{s}}, \boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})
\end{aligned}
$$

Linearization

$$
\begin{aligned}
\delta w^{\mathrm{s}}_{(2)} &= \int_{B^{\mathrm{s}}} \operatorname{grad} \delta \boldsymbol{v}^{\mathrm{s}} : \boldsymbol{\sigma}' dv \\
&= \int_{B^{\mathrm{s}}} \delta \boldsymbol{d}^{\mathrm{s}} : \boldsymbol{\sigma}' dv = \delta w^{\mathrm{s}}_{(2)}(\underline{\boldsymbol{u}^{\mathrm{s}}}) \\
\delta w^{\mathrm{s}}_{(3)} &= -\int_{B^{\mathrm{s}}} p^{f} \operatorname{div} \delta \boldsymbol{v}^{\mathrm{s}} dv = \delta w^{\mathrm{s}}_{(3)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{s}}_{(4)} &= -\int_{B^{\mathrm{s}}} \rho \delta \boldsymbol{v}^{\mathrm{s}} \cdot \mathbf{b} dv = \delta w^{\mathrm{s}}_{(4)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{s}}_{(5)} &= -\int_{\partial B^{\mathrm{s}}_{\mathrm{t}}} \delta \boldsymbol{v}^{\mathrm{s}} \cdot \bar{\mathbf{t}} da = \delta w^{\mathrm{s}}_{(5)}(t)
\end{aligned}
$$

$\delta \boldsymbol{v}^{\mathrm{s}}$: weight function

Weak forms (2)

- Mass and linear momentum balance of water

$$
\begin{aligned}
\delta w^{\mathrm{w}} =& \delta w^{\mathrm{w}}_{(1)} + \delta w^{\mathrm{w}}_{(2)} + \delta w^{\mathrm{w}}_{(3)} + \delta w^{\mathrm{w}}_{(4)} + \delta w^{\mathrm{w}}_{(5)} + \delta w^{\mathrm{w}}_{(6)} + \delta w^{\mathrm{w}}_{(7)} \\
\delta w^{\mathrm{w}}_{(1)} =& \int_{B^{\mathrm{s}}} \delta p^{\mathrm{w}} n \left(\frac{s^{\mathrm{w}} \rho^{\mathrm{wR}}}{K^{\mathrm{w}}} - \rho^{\mathrm{wR}} c \right) \dot{p}^{\mathrm{w}} dv = \delta w^{\mathrm{w}}_{(1)} (\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{w}}_{(2)} =& \int_{B^{\mathrm{s}}} \delta p^{\mathrm{w}} n \rho^{\mathrm{wR}} c \dot{p}^{\mathrm{a}} dv = \delta w^{\mathrm{w}}_{(2)} (\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{w}}_{(3)} =& \int_{B^{\mathrm{s}}} \delta p^{\mathrm{w}} s^{\mathrm{w}} \rho^{\mathrm{wR}} \operatorname{div} \boldsymbol{v}^{\mathrm{s}} dv = \delta w^{\mathrm{w}}_{(3)} (\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{w}}_{(4)} =& \int_{B^{\mathrm{s}}} \operatorname{grad} \delta p^{\mathrm{w}} \cdot \frac{\boldsymbol{k}^{\mathrm{ws}}}{g} \operatorname{grad} p^{\mathrm{w}} dv = \delta w^{\mathrm{w}}_{(4)} (\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{w}}_{(5)} =& -\int_{B^{\mathrm{s}}} \operatorname{grad} \delta p^{\mathrm{w}} \cdot \frac{\boldsymbol{k}^{\mathrm{ws}}}{g} \rho^{\mathrm{wR}} \boldsymbol{b} dv = \delta w^{\mathrm{w}}_{(5)} (\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{w}}_{(6)} =& \int_{B^{\mathrm{s}}} \operatorname{grad} \delta p^{\mathrm{w}} \cdot \frac{\boldsymbol{k}^{\mathrm{ws}}}{g} \rho^{\mathrm{wR}} \boldsymbol{a}^{\mathrm{s}} dv = \delta w^{\mathrm{w}}_{(6)} (\underline{\boldsymbol{u}^{\mathrm{s}}, \boldsymbol{a}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}}) \\
\delta w^{\mathrm{w}}_{(7)} =& \int_{\partial B^{\mathrm{s}}_{\mathrm{wq}}} \delta p^{\mathrm{w}} \bar{q}^{\mathrm{w}} da = \delta w^{\mathrm{w}}_{(7)} (t)
\end{aligned}
$$

Linearization

δp^{w} : weight function

Weak forms (3)

- Mass and linear momentum balance of air

$$\delta w^{\mathrm{a}} = \delta w^{\mathrm{a}}_{(1)} + \delta w^{\mathrm{a}}_{(2)} + \delta w^{\mathrm{a}}_{(3)} + \delta w^{\mathrm{a}}_{(4)} + \delta w^{\mathrm{a}}_{(5)} + \delta w^{\mathrm{a}}_{(6)} + \delta w^{\mathrm{a}}_{(7)}$$

$$\delta w^{\mathrm{a}}_{(1)} = \int \delta p^{\mathrm{a}} n \left(\frac{1 - s^{\mathrm{w}}}{\Theta \bar{R}} - \rho^{\mathrm{aR}} c \right) \dot{p}^{\mathrm{a}} dv = \delta w^{\mathrm{a}}_{(1)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})$$

Linearization

$$\delta w^{\mathrm{a}}_{(2)} = \int_{B^{\mathrm{s}}} \delta p^{\mathrm{a}} n \rho^{\mathrm{aR}} c \dot{p}^{\mathrm{w}} dv = \delta w^{\mathrm{a}}_{(2)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})$$

$$\delta w^{\mathrm{a}}_{(3)} = \int_{B^{\mathrm{s}}} \delta p^{\mathrm{a}} (1 - s^{\mathrm{w}}) \rho^{\mathrm{aR}} \operatorname{div} \boldsymbol{v}^{\mathrm{s}} dv = \delta w^{\mathrm{a}}_{(3)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})$$

$$\delta w^{\mathrm{a}}_{(4)} = \int_{B^{\mathrm{s}}} \operatorname{grad} \delta p^{\mathrm{a}} \cdot \frac{\boldsymbol{k}^{\mathrm{as}}}{g} \operatorname{grad} p^{\mathrm{a}} dv = \delta w^{\mathrm{a}}_{(4)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})$$

$$\delta w^{\mathrm{a}}_{(5)} = - \int_{B^{\mathrm{s}}} \operatorname{grad} \delta p^{\mathrm{a}} \cdot \frac{\boldsymbol{k}^{\mathrm{as}}}{g} \rho^{\mathrm{aR}} \boldsymbol{b} dv = \delta w^{\mathrm{a}}_{(5)}(\underline{\boldsymbol{u}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})$$

$$\delta w^{\mathrm{a}}_{(6)} = \int_{B^{\mathrm{s}}} \operatorname{grad} \delta p^{\mathrm{a}} \cdot \frac{\boldsymbol{k}^{\mathrm{as}}}{g} \rho^{\mathrm{aR}} \boldsymbol{a}^{\mathrm{s}} dv = \delta w^{\mathrm{a}}_{(6)}(\underline{\boldsymbol{u}^{\mathrm{s}}, \boldsymbol{a}^{\mathrm{s}}, p^{\mathrm{w}}, p^{\mathrm{a}}})$$

$$\delta w^{\mathrm{a}}_{(7)} = \int_{\partial B^{\mathrm{s}}_{\mathrm{aq}}} \delta p^{\mathrm{a}} \bar{q}^{\mathrm{a}} da = \delta w^{\mathrm{a}}_{(7)}(t)$$

δp^{a} : weight function

Weak forms

- Fully implicit with Newton-Raphson method

$$\delta w^{\mathrm{s}}(\boldsymbol{a}^{\mathrm{s}}, \ddot{p}^{\mathrm{w}}, \ddot{p}^{\mathrm{a}}) = 0 \quad \boldsymbol{a}^{\mathrm{s}}_{k+1} = \boldsymbol{a}^{\mathrm{s}}_{k} + \Delta \boldsymbol{a}^{\mathrm{s}}$$

$$\delta w^{\mathrm{w}}(\boldsymbol{a}^{\mathrm{s}}, \ddot{p}^{\mathrm{w}}, \ddot{p}^{\mathrm{a}}) = 0 \quad \ddot{p}^{\mathrm{w}}_{k+1} = \ddot{p}^{\mathrm{w}}_{k} + \Delta \ddot{p}^{\mathrm{w}}$$

$$\delta w^{\mathrm{a}}(\boldsymbol{a}^{\mathrm{s}}, \ddot{p}^{\mathrm{w}}, \ddot{p}^{\mathrm{a}}) = 0 \quad \ddot{p}^{\mathrm{a}}_{k+1} = \ddot{p}^{\mathrm{a}}_{k} + \Delta \ddot{p}^{\mathrm{a}}$$

$$D\delta w^{\mathrm{s}}[\Delta \boldsymbol{a}^{\mathrm{s}}] + D\delta w^{\mathrm{s}}[\Delta \ddot{p}^{\mathrm{w}}] + D\delta w^{\mathrm{s}}[\Delta \ddot{p}^{\mathrm{a}}] = -\delta w^{\mathrm{s}}_{k}$$

$$D\delta w^{\mathrm{w}}[\Delta \boldsymbol{a}^{\mathrm{s}}] + D\delta w^{\mathrm{w}}[\Delta \ddot{p}^{\mathrm{w}}] + D\delta w^{\mathrm{w}}[\Delta \ddot{p}^{\mathrm{a}}] = -\delta w^{\mathrm{w}}_{k}$$

$$D\delta w^{\mathrm{a}}[\Delta \boldsymbol{a}^{\mathrm{s}}] + D\delta w^{\mathrm{a}}[\Delta \ddot{p}^{\mathrm{w}}] + D\delta w^{\mathrm{a}}[\Delta \ddot{p}^{\mathrm{a}}] = -\delta w^{\mathrm{a}}_{k}$$

Linearization (the first term in Linear momentum balance of mixture)

- Linearization of weak forms at reference configuration obtained by pull back

$$
\begin{aligned}
D\delta W^{\mathrm{s}}_{(1)}[\Delta \boldsymbol{a}^{\mathrm{s}}] &= \int_{B^{s0}} \left(D\rho_{s0}[\Delta \boldsymbol{a}^{\mathrm{s}}]\ \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} + \rho_{s0} \delta \boldsymbol{v}^{\mathrm{s}} \cdot D\boldsymbol{a}^{\mathrm{s}}[\Delta \boldsymbol{a}^{\mathrm{s}}] \right) dV^{\mathrm{s}} \\
&= \int_{B^{s0}} \left\{ \beta \Delta t^2 J^{\mathrm{s}} \left(s^{\mathrm{w}} \rho^{\mathrm{wR}} + s^{\mathrm{a}} \rho^{\mathrm{aR}} \right) \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}}\ \operatorname{div} \Delta \boldsymbol{a}^{\mathrm{s}} \right\} dV^{\mathrm{s}} \\
&\quad + \int_{B^{s0}} \left(\rho_{s0} \delta \boldsymbol{v}^{\mathrm{s}} \cdot \Delta \boldsymbol{a}^{\mathrm{s}} \right) dV^{\mathrm{s}}
\end{aligned}
$$

$$
\begin{aligned}
D\delta W^{\mathrm{s}}_{(1)}[\Delta \ddot{p}^{\mathrm{w}}] &= \int_{B^{s0}} D\rho_{s0}[\Delta \ddot{p}^{\mathrm{w}}] \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} dV^{\mathrm{s}} \\
&= \int_{B^{s0}} \beta \Delta t^2 (J^{\mathrm{s}} - 1 + n_{s0}) \left\{ \frac{s^{\mathrm{w}} \rho^{\mathrm{wR}}}{K^{\mathrm{w}}} + c(\rho^{\mathrm{aR}} - \rho^{\mathrm{wR}}) \right\} \Delta \ddot{p}^{\mathrm{w}} \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} dV^{\mathrm{s}}
\end{aligned}
$$

$$
\begin{aligned}
D\delta W^{\mathrm{s}}_{(1)}[\Delta \ddot{p}^{\mathrm{a}}] &= \int_{B^{s0}} D\rho_{s0}[\Delta \ddot{p}^{\mathrm{a}}]\ \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} dV^{\mathrm{s}} \\
&= \int_{B^{s0}} \beta \Delta t^2 (J^{\mathrm{s}} - 1 + n_{s0}) \left\{ \frac{s^{\mathrm{a}}}{\Theta \bar{R}} - c(\rho^{\mathrm{aR}} - \rho^{\mathrm{wR}}) \right\} \Delta \ddot{p}^{\mathrm{a}} \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} dV^{\mathrm{s}}
\end{aligned}
$$

Linearization (the first term in Linear momentum balance of mixture)

- Push forward of linearized weak forms to current configuration

$$D\delta w^{\mathrm{s}}_{(1)}[\Delta \boldsymbol{a}^{\mathrm{s}}] = \beta \Delta t^2 \int_{B^{\mathrm{s}}} (s^{\mathrm{w}} \rho^{\mathrm{wR}} + s^{\mathrm{a}} \rho^{\mathrm{aR}}) \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} \; \mathrm{div}\, \Delta \boldsymbol{a}^{\mathrm{s}} dv + \int_{B^{\mathrm{s}}} \rho \delta \boldsymbol{v}^{\mathrm{s}} \cdot \Delta \boldsymbol{a}^{\mathrm{s}} dv$$

$$D\delta w^{\mathrm{s}}_{(1)}[\Delta \ddot{p}^{\mathrm{w}}] = \beta \Delta t^2 \int_{B^{\mathrm{s}}} n \left\{ \frac{s^{\mathrm{w}} \rho^{\mathrm{wR}}}{K^{\mathrm{w}}} + c(\rho^{\mathrm{aR}} - \rho^{\mathrm{wR}}) \right\} \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} \Delta \ddot{p}^{\mathrm{w}} dv$$

$$D\delta w^{\mathrm{s}}_{(1)}[\Delta \ddot{p}^{\mathrm{a}}] = \beta \Delta t^2 \int_{B^{\mathrm{s}}} n \left\{ \frac{s^{\mathrm{a}}}{\Theta \bar{\bar{R}}} - c(\rho^{\mathrm{aR}} - \rho^{\mathrm{wR}}) \right\} \delta \boldsymbol{v}^{\mathrm{s}} \cdot \boldsymbol{a}^{\mathrm{s}} \Delta \ddot{p}^{\mathrm{a}} dv$$

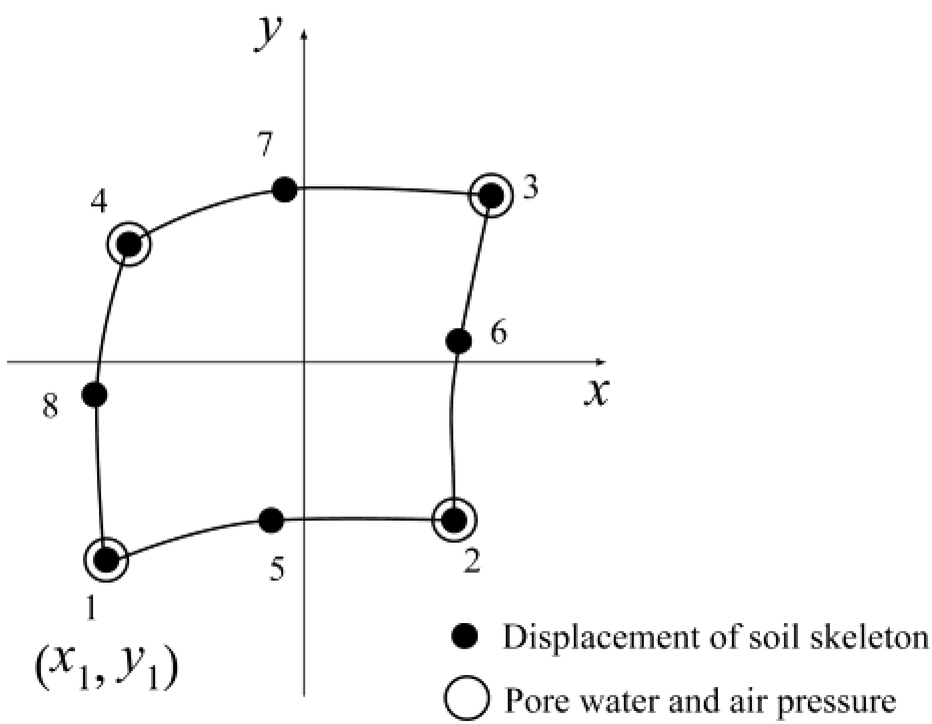
Mixed finite element method

- 8-node isoparametric element
  - Satisfy LBB condition
  - Soil skeleton: 8 nodes
  - Pore water and air: 4nodes

$$\boldsymbol{u}_e^{\mathrm{s}} = \sum_{n=1}^{8} N_n^{\mathrm{s}} \boldsymbol{u}_n^{\mathrm{s}}$$

$$p_e^{\mathrm{w}} = \sum_{n=1}^{4} N_n^{\mathrm{f}} p_n^{\mathrm{w}}$$

$$p_e^{\mathrm{a}} = \sum_{n=1}^{4} N_n^{\mathrm{f}} p_n^{\mathrm{a}}$$



Time integration

- Newmark implicit time integration method

$$\boldsymbol{u}^{\mathrm{s}} = \boldsymbol{u}_t^{\mathrm{s}} + \Delta t\, \boldsymbol{v}_t^{\mathrm{s}} + \frac{1}{2}\Delta t^2 (1-2\beta)\, \boldsymbol{a}_t^{\mathrm{s}} + \beta \Delta t^2\, \boldsymbol{a}^{\mathrm{s}}$$

$$\boldsymbol{v}^{\mathrm{s}} = \boldsymbol{v}_t^{\mathrm{s}} + (1-\gamma)\Delta t\, \boldsymbol{a}_t^{\mathrm{s}} + \gamma \Delta t\, \boldsymbol{a}^{\mathrm{s}}$$

$$p^{\mathrm{w}} = p_t^{\mathrm{w}} + \Delta t\, \dot{p}_t^{\mathrm{w}} + \frac{1}{2}\Delta t^2 (1-2\beta)\, \ddot{p}_t^{\mathrm{w}} + \beta \Delta t^2\, \ddot{p}^{\mathrm{w}}$$

$$\dot{p}^{\mathrm{w}} = \dot{p}_t^{\mathrm{w}} + (1-\gamma)\Delta t\, \ddot{p}_t^{\mathrm{w}} + \gamma \Delta t\, \ddot{p}^{\mathrm{w}}$$

$$p^{\mathrm{a}} = p_t^{\mathrm{a}} + \Delta t\, \dot{p}_t^{\mathrm{a}} + \frac{1}{2}\Delta t^2 (1-2\beta)\, \ddot{p}_t^{\mathrm{a}} + \beta \Delta t^2\, \ddot{p}^{\mathrm{a}}$$

$$\dot{p}^{\mathrm{a}} = \dot{p}_t^{\mathrm{a}} + (1-\gamma)\Delta t\, \ddot{p}_t^{\mathrm{a}} + \gamma \Delta t\, \ddot{p}^{\mathrm{a}}$$